REVIEW.

1) Three Fundamental Theorems

(a) Open Mapping Theorem

X, Y B-spaces $A \in \mathcal{L}(X,Y)$ onto $\Rightarrow A(\cdot)$ is open

$\Downarrow$

(a_1) If A is also 1-1, then $A^{-1} \in \mathcal{L}(Y,X)$ (isomorphism)

(a_2) Y is isomorphic to $X/_{N(A)}$

(a_3) $\exists\, M > 0$ s.t $\forall y \in Y$ $\exists\, x \in A^{-1}(y)$ s.t $\|x\|_X \leq M\|y\|_Y$

(b) Closed Graph Theorem

X, Y B-spaces, $A : X \to Y$ linear with closed graph

$\Downarrow$

$A \in \mathcal{L}(X,Y)$

(c) Uniform Boundedness Principle (Banach-Steinhaus)

X, Y B-spaces, $\mathcal{A} \subseteq \mathcal{L}(X,Y)$ pointwise bounded

$(\sup\{\|A(x)\|_Y : A \in \mathcal{A}\} < \infty)$

$\Downarrow$

$\sup\{\|A\|_{\mathcal{L}} : A \in \mathcal{A}\} < \infty$.

(c_1) $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}(X,Y)$, $A_n(x) \to A(x)$ $\forall x \in X$

$\Downarrow$

$A \in \mathcal{L}(X,Y)$, $\|A\|_{\mathcal{L}} \leq \liminf_{n\to\infty} \|A_n\|_{\mathcal{L}}$

(c_2) $D \subseteq X$ w-bounded $(\sup\{|\langle x^*, u\rangle| : u \in D\} < \infty\}$ $\forall x^* \in X^*)$

$\Downarrow$

D bounded

(C3) $D^* \subseteq X^*$ is w^* bounded $\left(\sup\{|\langle u^*, x\rangle| : u^* \in D^*\} < \infty\right)\ \forall x \in X$

$$\Downarrow$$

D bounded

2) Weak and Weak* Topologies.

τ = topology on X $\quad (X_\tau)^* = X^*$

weak topology = weakest such topology.

Basic element at the origin

$\mathcal{U} = \{x \in X : |\langle x_k^*, x\rangle| < \varepsilon,\ k = 1, \dots, n\}$

$\forall \{x_1^*, \dots, x_n^*\} \subseteq X^*,\ \forall \varepsilon > 0$

τ = topology on X^* $\quad (X^*_\tau)^* = X$

weak* topology = weakest such topology

$\mathcal{U} = \{x^* \in X^* : |\langle x^*, x_k\rangle| < \varepsilon,\ k = 1, \dots, n\}$

$\forall \{x_1, \dots, x_n\} \subseteq X,\ \forall \varepsilon > 0$

$w \subseteq s$ in X $\quad , \quad w^* \subseteq w \subseteq s$ in X^*

If X = infinite dimensional and $U \subseteq X$ w-open

then $\mathcal{U}$ is not bounded

So $w \neq s$

Theorem (Mazur)

If X is a Banach space and $C \subseteq X$ convex

then $\overline{C}^w = \overline{C}$

Remark: This theorem fails for the w^*-topology.

"If $u_n \xrightarrow{w} u$ in X

then $\exists$ convex combinations y_n of $\{u_n\}_{n\in\mathbb{N}}$ s.t. $y_n \longrightarrow u$

Theorem (Alaoglu)

$\overline{B}_1^* = \{x^* \in X^* : \|x^*\|_* \leq 1\}$ is w^*-compact.

3) Separable and Reflexive B-Spaces.

$i: X \longrightarrow X^{**}$ $\quad i(x)(u^*) = \langle u^*, x\rangle$ $\quad \forall x \in X, \forall u^* \in X^*$

"Canonical Embedding"

(a) Separable : $\exists D \subseteq X$ countable $\overline{D} = X$

(b) Reflexive: Canonical embedding is surjective

Theorem (Goldstine)

$\overline{B_1^X}^{w^*} = \overline{B}_1^{X^{**}}$ $\quad$ (so $\overline{\partial B_1^X}^{w^*} = \partial B_1^{X^{**}}$, $\overline{X}^{w^*} = X^{**}$)

Metrizability of the w-topology

(i) $(\overline{B}_1^X, w)$ metrizable iff X^* is separable

(ii) $(\overline{B}_1^{X^*}, w^*)$ metrizable iff X is separable

(iii) If X = separable, $K \subseteq X$ w-compact,

then (K, w) = metrizable

X^* separable $\Rightarrow$ X separable; the converse not true

Theorem (James 1)

If $X = B$-space, $K \subseteq X$ w-closed and every $x^* \in X^*$ attains its supremum on K ($\sup[\langle x^*, u\rangle : u \in K]$ is realized)

then K is w-compact

For reflexive B-spaces $X = X^{**}$ and so $w = w^*$

Theorem (James 2)

X is reflexive iff $\overline{B}_1^X$ is w-compact

X Reflexive $\Leftrightarrow$ $X^* =$ Reflexive

So $X =$ Separable, reflexive $\Leftrightarrow$ $X^* =$ Separable, reflexive

$X =$ Separable, reflexive $\Rightarrow$ $(\overline{B}_1^X, w)$ compact metrizable

$X =$ Reflexive, $V \subseteq X$ closed subspace $\Rightarrow$ $V =$ Reflexive

$X = B$-space, $K \subseteq X$ compact (resp. w-compact)

$\Downarrow$

$\overline{\text{conv}}\, K$ is compact (resp w-compact)

(Krein-Smulian (resp Mazur))

Theorem (Eberlein-Smulian).

$C \subseteq X$ is (relatively) w-compact

iff

C is (relatively) sequentially w-compact

X, Y B-spaces $\quad A \in \mathcal{L}(X,Y)$

Dual (adjoint) operator $\quad A^* \in \mathcal{L}(Y^*, X^*)$

$$\langle A^*(y^*), x \rangle_X = \langle y^*, A(x) \rangle_Y$$

$$\Downarrow$$

$$\| A \|_{\mathcal{L}} = \| A^* \|_{\mathcal{L}}$$

$A \in \mathcal{L}(X,Y)$ compact, if $A(\bar{B}_1^X)$ is relatively compact in Y

$$\mathcal{L}_c(X,Y)$$

$A \in \mathcal{L}(X,Y)$ finite rank, if $\dim R(A) < \infty$

$$\mathcal{L}_f(X,Y)$$

$\mathcal{L}_c(X,Y)$ a closed subspace of $\mathcal{L}(X,Y)$, thus a Banach space

$A \in \mathcal{L}(X,Y)$ completely continuous, if $x_n \xrightarrow{w} x \Rightarrow A(x_n) \to A(x)$.

$A \in \mathcal{L}_c(X,Y) \Rightarrow A$ is completely continuous

X = Reflexive, $A \in \mathcal{L}(X,Y)$ is completely continuous

$$\Downarrow$$

$$A \in \mathcal{L}_c(X,Y)$$

Theorem (Schauder)

$A \in \mathcal{L}(X,Y)$. Then we have

$A \in \mathcal{L}_c(X,Y)$ iff $A^* \in \mathcal{L}_c(Y^*, X^*)$.

If $A \in \mathcal{L}_c(X)$ and $\lambda \neq 0$,

then $N(A - \lambda I)$ is finite dimensional,

$(A - \lambda I)(X)$ is closed and finite codimensional.

(dim $(X/R(A)) < \infty$)

Dfn: $A \in \mathcal{L}(X,Y)$ is a "Fredholm operator"

iff

$N(A)$ is finite dimensional

$R(A)$ is finite codimensional.

$i(A) = \dim N(A) - \operatorname{codim} R(A)$. $\longrightarrow$ index of A

Dfn $A \in \mathcal{L}(X,Y)$ is invertible, if A is an isomorphism of X onto Y

$A \in \mathcal{L}(X)$ and $\|A\|_{\mathcal{L}} < 1$

$\Downarrow$

$(I - A)$ is invertible, $(I-A)^{-1} = \sum_{k \geq 0} A^k$

So, the set of invertible operators $A \in \mathcal{L}(X)$ is open

$A \longrightarrow A^{-1}$ homeomorphism.

Dfn: $A \in \mathcal{L}(X)$ $\quad \sigma(A) = \{ \lambda \in F : \lambda I - A$ is not invertible $\}$

$\downarrow$

Spectrum of A

$\rho(A) = F \setminus \sigma(A)$ resolvent set (regular values)

$\lambda \in \rho(A) \Rightarrow (\lambda I - A)^{-1}$ = resolvent of A at λ

Remark: If X = infinite dimensional and $A \in \mathcal{L}_c(X)$, then $0 \in \sigma(A)$.

$\sigma(A) \subseteq F$ compact set in $\overline{B}_{\|A\|_{\mathcal{L}}}$

$\sigma(A) \neq \emptyset$ if $F = \mathbb{C}$

__Dfn:__ $A \in \mathcal{L}(X)$ λ = __eigenvalue__ iff $N(\lambda I - A) \neq \{0\}$

$x \in N(\lambda I - A) \setminus \{0\}$ __eigenvector__.

$N(\lambda I - A)$ = __eigenspace__

Eigenvectors corresponding to distinct eigenvalues are linearly independent

$A \in \mathcal{L}_c(X)$, $\lambda \in \sigma(A) \setminus \{0\} \Rightarrow \lambda$ = eigenvalue of A

$A \in \mathcal{L}_c(X) \Rightarrow$ the eigenvalues have only 0 as limit point.

4) Hilbert Spaces.

H = Hilbert space

__Riesz-Frechet Thm:__ $H^* = H$

$A \in \mathcal{L}(H)$. The "__adjoint operator__" $A^* \in \mathcal{L}(H)$ is defined by

$$(A(u), h) = (u, A^*(h)) \quad \forall u, h \in H.$$

$$\lambda \in \sigma(A) \text{ iff } \overline{\lambda} \in \sigma(A^*).$$

$$N(A^*) = R(A)^{\perp} \qquad N(A) = R(A^*)^{\perp}$$

__Dfn__ $A \in \mathcal{L}(H)$ is "self-adjoint" (s.a) iff $A^* = A$.

$$(A(u), h) = (u, A(h)) \quad \forall u, h \in H$$

Parallelogram law $\|x+u\|^2 + \|x-u\|^2 = 2\|x\|^2 + 2\|u\|^2$

Polarization identity.

$$(x,u) = \frac{1}{4}\left[\|x+u\|^2 - \|x-u\|^2\right] \quad \text{(Real case)}$$

$$(x,u) = \frac{1}{4}\left[\|x+u\|^2 - \|x-u\|^2 + i\|x+iu\|^2 - i\|x-iu\|^2\right]$$

(Complex case)

(i) $A \in \mathcal{L}(H)$ s.a $\Rightarrow$ $\|A\|_{\mathcal{L}} = \sup\left[\,|(A(u),u)| : \|u\| \leq 1\right]$

$(A(u),u) \in \mathbb{R}$, Eigenvalues are in $\mathbb{R}$

$\|A^n\|_{\mathcal{L}} = \|A\|_{\mathcal{L}}^n \quad \forall n \in \mathbb{N}$

(ii) If $A \in \mathcal{L}(H)$ s.a and $\lambda \in \mathbb{R}$, then $\lambda \in \sigma(A)$ iff $\inf_{\|x\|=1} \|(\lambda I - A)(x)\| = 0$

(iii) If $A \in \mathcal{L}(H)$ s.a, $m_A = \inf_{\|u\|=1} (A(u),u)$, $M_A = \sup_{\|u\|=1} (A(u),u)$,

then $\sigma(A) \subseteq [m_A, M_A]$ $\quad m_A, M_A \in \sigma(A)$.

(iv) $A \in \mathcal{L}(H)$ s.a $\Rightarrow$ eigenvectors corresponding to distinct eigenvalues are orthogonal

Theorem: If H = infinite dimensional Hilbert space, $A \in \mathcal{L}_c(H)$ s.a,

then $\sigma(A) = \{0\} \cup \{\lambda_k\}$, λ_k = distinct eigenvalues.

$\|A\|_{\mathcal{L}} \in \{\lambda_k\}$ and $\{\lambda_k\}$ is either finite or a sequence converging to 0.

Theorem: If H = infinite dimensional separable Hilbert space then there is an o.n. basis $\{e_n\}$ of H such that each e_n = eigenvector corresponding to some real eigenvalue λ_n of A and for all $x \in H$

$$A(x) = \sum_{k \geq 1} \lambda_k (x, e_k) e_k$$

Dfn: $A \in \mathcal{L}(H)$

(a) Normal $AA^* = A^*A$

(b) Unitary $AA^* = A^*A = I$. $(A^* = A^{-1})$.

(c) Projection $A^2 = A$.

Unitary $\Rightarrow$ Normal, s.a $\Rightarrow$ Normal

$A \in \mathcal{L}(H)$ normal $\Leftrightarrow \|A(u)\| = \|A^*(u)\| \quad \forall u \in H$.

$A \in \mathcal{L}(H)$ normal $\Rightarrow N(A) = N(A^*) = R(A)^{\perp}$

$A \in \mathcal{L}(H)$ normal and λ, μ distinct eigenvalues

$\Downarrow$

corresponding eigenspaces orthogonal

$H = N(A) \oplus R(A)$.

Dfn: $P \in \mathcal{L}(H)$ projection, it is an "orthogonal projection"

iff

$N(P) \perp R(P)$

Equivalently $R(P) = N(P)^{\perp} \qquad N(P) = R(P)^{\perp}$

If $P \in \mathcal{L}(H)$ projection,

then the following are equivalent

(a) P = orthogonal projection;

(b) P is s.a

(c) P is normal

(d) $(u - P(u), P(u)) = 0 \quad \forall u \in H$.

(e) $\|P\|_{\mathcal{L}} = 1$

5) Lebesgue Spaces

$\Omega \subseteq \mathbb{R}^N$ bounded open.

$1 \leq p < \infty \quad L^p(\Omega) = \{u : \int_\Omega |u|^p dz < \infty\} \quad \|u\|_p = \left(\int_\Omega |u|^p dz\right)^{1/p}$

$p = \infty \quad L^\infty(\Omega) = \{u : \exists M > 0 \quad |u(z)| \leq M \text{ a.e.}\}$

$\|u\|_\infty = \inf\{M : |\{|u| > M\}|_N = 0\}$.

Banach spaces

$L^p(\Omega) \quad 1 \leq p < \infty$ separable

$L^p(\Omega) \quad 1 < p < \infty$ reflexive (uniformly convex)

$(\forall \varepsilon > 0 \ \exists \delta > 0$ s.t $\|u\| = \|x\| = 1$

$\|x - u\| \geq \varepsilon \Rightarrow \|\frac{1}{2}(x+u)\| \leq 1 - \delta)$

Hölder inequality: $\frac{1}{p} + \frac{1}{p'} = 1 \quad u \in L^p(\Omega),\ h \in L^{p'}(\Omega)$

$\Downarrow$

$uh \in L^1(\Omega) \quad \|uh\|_1 \leq \|u\|_p \|h\|_{p'}$

If $1 \leq p \leq q \leq \infty$, then $L^q(\Omega) \hookrightarrow L^p(\Omega)$

Interpolation Inequality: $1 \leq p \leq q \leq \infty$ $u \in L^p \cap L^q$ (Ω unbounded)

$$\Downarrow$$

$u \in L^r(\Omega)$, $p \leq r \leq q$ $\|u\|_r \leq \|u\|_p^t \|u\|_q^{1-t}$

$$\frac{1}{r} = \frac{t}{p} + \frac{1-t}{q}$$

$L^p(\Omega)^* = L^{p'}(\Omega)$ $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$

$\langle u, g \rangle = \int_\Omega u g \, dz$. (Riesz Representation Theorem)

Dfn: $C \subseteq L^1(\Omega)$ "uniformly integrable"

iff

bounded and $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t

$$|A|_N \leq \delta \Rightarrow \sup_{u \in C} \int_A |u| \, dz \leq \varepsilon$$

iff

$$\lim_{\lambda \downarrow 0} \sup_{u \in C} \int_{\{|u| \geq \lambda\}} |u| \, dz = 0.$$

Theorem (Dunford - Pettis)

$C \subseteq L^1(\Omega)$ is relatively w-compact

iff

it is uniformly integrable

If $\dim X < \infty$ or $\dim Y < \infty$ and $A \in \mathcal{L}(X,Y)$

then A = surjective iff A^* = injective,

A^* = surjective iff A = injective

In the general case we have

A = surjective $\Rightarrow$ A^* = injective,

A^* = surjective $\Rightarrow$ A = injective.