

Algorithmic Game Theory

Introduction to Mechanism Design for Single Parameter Environments

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Mechanism Design

- What is mechanism design?
- It can be seen as **reverse game theory**
- **Main goal:** design the rules of a game so as to
 - deal with strategic behavior by the players
 - **strategic behavior:** declaring false preferences in order to gain a higher utility
 - design rules (i.e., a strategic game) so as to enforce certain equilibrium among players (by exploiting strategic behavior)
- Applied to problems where a “social choice” needs to be made
 - i.e., an aggregation of individual preferences to a single joint decision

Examples

- Elections

- Parliamentary elections, committee elections, council elections, etc
- A set of voters
- A set of candidates
- Each voter expresses preferences according to the election rules
 - E.g., by specifying his single top choice, or by specifying his first few choices, or by submitting a full ranking of the candidates
- **Social choice:** can be a single candidate (single-winner election) or a set of candidates (multi-winner election) or a ranking of the candidates

Examples

- Government policy making and referenda
 - A municipality is considering implementing a public project
 - Q1: Should we build a new road, a library or a tennis court?
 - Q2: If we build a library, where shall we build it?
 - Citizens can express their preferences in an online survey or a referendum
 - **Social choice:** the decision of the municipality on what and where to implement

Examples

- Auctions

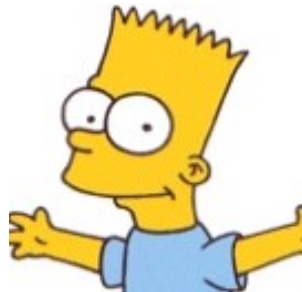
- An auctioneer with some items for sale
- A set of bidders express preferences (offers) over items
 - Or combinations of items
- Preferences are submitted either through a valuation function, or according to some bidding language
- **Social choice:** allocation of items to the bidders

Specifying preferences

- In all the examples, the players need to submit their preferences in some form
- Representation of preferences can be done by
 - A valuation function (specifying a value for each possible outcome)
 - A ranking (an ordering on possible outcomes)
 - An approval set (which outcomes are approved)
- Possible conflict between **increased expressiveness** vs **complexity of decision problem**

Single-item Auctions

Auctions



Set of players
 $N = \{1, 2, \dots, n\}$



1 indivisible good

Auctions

- A means of conducting transactions since antiquity
 - First references of auctions date back to ancient Athens and Babylon
- Modern applications:
 - Art works
 - Stamps
 - Flowers (Netherlands)
 - Spectrum licenses
 - Other governmental licenses
 - Pollution rights
 - Google ads
 - eBay
 - Bonds
 - ...

Auctions

- Earlier, the most popular types of auctions were
 - The English auction
 - The price keeps increasing in small increments
 - Gradually bidders drop out till there is only one winner left
 - The Dutch auction
 - The price starts at $+\infty$ (i.e., at some very high price) and keeps decreasing
 - Until there exists someone willing to offer the current price
 - There exist also many variants regarding their practical implementation
- These correspond to ascending or descending price trajectories

Sealed bid auctions

- **Sealed bid:** We think of every bidder submitting his bid in an envelope, without other players seeing it
 - It does not really have to be an envelope, bids can be submitted electronically
 - The main assumption is that it is submitted in a way that other bidders cannot see it
- **After collecting the bids, the auctioneer needs to decide:**
 - Who wins the item?
 - Easy! Should be the guy with the highest bid
 - Yes in most cases, but not always
 - How much should the winner pay?
 - Not so clear

Sealed bid auctions

Why do we view auctions as games?

- We assume every player has a valuation v_i for obtaining the good
- **Available strategies:** each bidder is asked to submit a bid b_i
 - $b_i \in [0, \infty)$
 - Infinite number of strategies
- The submitted bid b_i may differ from the real value v_i of bidder i

First price auction

Auction rules

- Let $\mathbf{b} = (b_1, b_2, \dots, b_n)$ the vector of all the offers
- **Winner:** The bidder with the highest offer
 - In case of ties: We assume the winner is the bidder with the lowest index (not important for the analysis)
 - E.g. if there is a tie among bidder 2 and bidder 4, the winner is bidder 2
- **Winner's payment:** the bid declared by the winner
- Utility function of bidder i ,

$$u_i(\mathbf{b}) = \begin{cases} v_i - b_i, & \text{if } i \text{ is the winner} \\ 0, & \text{otherwise} \end{cases}$$

Incentives in the first price auction

Analysis of first price auctions

- There are too many Nash equilibria
- Can we predict bidding behavior? Is some equilibrium more likely to occur?
- Hard to tell what exactly will happen in practice but we can still make some conclusions for first price auctions

Observation: Suppose that $v_1 \geq v_2 \geq v_3 \dots \geq v_n$. Then the profile $(v_2, v_2, v_3, \dots, v_n)$ is a Nash equilibrium

Corollary: The first price auction provides incentives to bidders to hide their true value

- This is highly undesirable when $v_1 - v_2$ is large

Auction mechanisms

We would like to explore alternative payment rules with better properties

Definition: For the single-item setting, an **auction mechanism** receives as input the bidding vector $\mathbf{b} = (b_1, b_2, \dots, b_n)$ and consists of

- an **allocation algorithm** (who wins the item)
- a **payment algorithm** (how much does the winner pay)

Most mechanisms satisfy **individual rationality:**

- Non-winners do not pay anything
- If the winner is bidder i , her payment will not exceed b_i (it is guaranteed that no-one will pay more than what she declared)

Auction mechanisms

Aligning Incentives

- Ideally, we would like mechanisms that do not provide incentives for strategic behavior
- How do we even define this mathematically?

An attempt:

Definition: A mechanism is called **truthful (or strategyproof, or incentive compatible)** if for every bidder i , and **for every profile** \mathbf{b}_{-i} of the other bidders, it is a **dominant strategy** for i to declare her real value v_i , i.e., it holds that

$$u_i(v_i, \mathbf{b}_{-i}) \geq u_i(b', \mathbf{b}_{-i}) \text{ for every } b' \neq v_i$$

Auction mechanisms

- In a truthful mechanism, every rational agent knows what to play, independently of what the other bidders are doing
- It is a win-win situation:
 - The auctioneer knows that players should not strategize
 - The bidders also know that they should not spend time on trying to find a different strategy
- Very powerful property for a mechanism
- **Fact:** The first-price mechanism is not truthful

Are there truthful mechanisms?

The 2nd price mechanism (Vickrey auction)

[Vickrey '61]

- **Allocation algorithm:** same as before, the bidder with the highest offer
 - In case of ties: we assume the winner is the bidder with the lowest index
- **Payment algorithm:** the winner pays the 2nd highest bid
- Hence, the auctioneer offers a discount to the winner

Observation: the payment does not depend on the winner's bid!

- The bid of each player determines if he wins or not, but not what he will pay

The 2nd price mechanism (Vickrey auction)

[Vickrey '61] (Nobel prize in economics, 1996)

• **Theorem:** The 2nd price auction is a truthful mechanism

Proof sketch:

• Fix a bidder i , and let \mathbf{b}_{-i} be an arbitrary bidding profile for the rest of the players

• Let $b^* = \max_{j \neq i} b_j$

• Consider now all possible cases for the final utility of bidder i , if he plays b_i

- $b_i < b^*$
- $b_i > b^*$
- $b_i = b^*$
- In all these different cases, we can prove that bidder i does not become better off by deviating to another strategy

Optimization objectives

What do we want to optimize in an auction?

Usual objectives:

- **Social welfare** (the total welfare produced for the involved entities)
- **Revenue** (the payment received by the auctioneer)

We will focus first on social welfare

Optimization objectives

What do we want to optimize in an auction?

Definition: The utilitarian social welfare produced by a bidding vector \mathbf{b} is $SW(\mathbf{b}) = \sum_i u_i(\mathbf{b})$

- The summation includes the auctioneer's utility (= the auctioneer's payment)
- The auctioneer's payment cancels out with the winner's payment

➤ For the single-item setting, $SW(\mathbf{b})$ = the value of the winner for the item

➤ An auction is **welfare maximizing** if it always produces an allocation with optimal social welfare when bidders are truthful

Vickrey auction: an ideal auction format

Summing up:

Theorem: The 2nd price auction is

- truthful [incentive guarantees]
- welfare maximizing [economic performance guarantees]
- implementable in polynomial time [computational performance guarantees]

Even though the valuations are private information to the bidders, the Vickrey auction solves the welfare maximization problem as if the valuations were known

Generalizations to single-parameter environments

Single-parameter mechanisms

- In many cases, we do not have a single item to sell, but multiple items
- But still, the valuation of a bidder could be determined by a single number (e.g., value per unit)
- **Note:** the valuation function may depend on various other parameters, but we assume only a single parameter is private information to the bidder
 - The **other parameters** may be **publicly known information**
- We can treat all these settings in a unified manner
- Our focus: **Direct revelation mechanisms**
 - The mechanism asks each bidder to submit the parameter that completely determines her valuation function

Examples of single-parameter environments

- **Single-item auctions:**

- One item for sale
- each bidder is asked to submit his value for acquiring the item

- **k-item unit-demand auctions**

- k identical items for sale
- each bidder submits his value per unit and can win at most one unit

- **Knapsack auctions**

- k identical items, each bidder has a value for obtaining a certain number of units

- **Single-minded auctions**

- a set of (non-identical) items for sale
- each bidder is interested in acquiring a specific subset of items (known to the mechanism)
- Each bidder submits his value for the set she desires

Examples of single-parameter environments

- **Sponsored search auctions**

- multiple advertising slots available, arranged from top to bottom
- each bidder interested in acquiring as high a slot as possible
- each bidder submits his value per click

- **Public project mechanisms**

- deciding whether to build a public project (e.g., a park)
- each bidder submits his value for having the project built

In all these settings, we can have multiple winners in the auction

Some Notation

- Suppose we have n players
- Let v_i be the parameter that is private information to player i
 - Usually v_i corresponds to value per unit, or value obtained at the desirable outcome, or maximum amount willing to pay (dependent on the context)

General form of direct-revelation mechanisms for single-parameter problems:

- **Input:** The bidding vector $\mathbf{b} = (b_1, \dots, b_n)$ by the players
 - each b_i may differ from v_i
- **Allocation rule:** Choose an allocation $\mathbf{x}(\mathbf{b}) = (x_1(\mathbf{b}), x_2(\mathbf{b}), \dots, x_n(\mathbf{b}))$
 - $x_i(\mathbf{b})$ = number of units received by pl. i or more generally the decision on what is allocated to i
- **Payment rule:** $\mathbf{p}(\mathbf{b}) = (p_1(\mathbf{b}), p_2(\mathbf{b}), \dots, p_n(\mathbf{b}))$
 - $p_i(\mathbf{b})$ = payment for bidder i

Some Notation

- We will use (\mathbf{x}, \mathbf{p}) to refer to a mechanism with allocation function \mathbf{x} , and payment function \mathbf{p}
- Final utility of bidder i in a mechanism $M = (\mathbf{x}, \mathbf{p})$:
 - $u_i(\mathbf{b}) = v_i x_i(\mathbf{b}) - p_i(\mathbf{b})$
 - Quasi-linear form of utility functions
- For simplicity, we often write (x_1, x_2, \dots, x_n) instead of $(x_1(\mathbf{b}), x_2(\mathbf{b}), \dots, x_n(\mathbf{b}))$
- We focus on mechanisms that satisfy **Individual Rationality**:
 - If a bidder i is a non-winner ($x_i(\mathbf{b}) = 0$), then $p_i(\mathbf{b}) = 0$
 - For winners, the payment rule satisfies $p_i(\mathbf{b}) \in [0, b_i x_i(\mathbf{b})]$ for every bidding vector \mathbf{b} and every i
 - The auctioneer can never ask a bidder for a payment higher than her declared total value for what she won

Examples of single-parameter environments

Describing the feasible allocations

- **Single-item auctions:**

- $x_i \in \{0, 1\}$ for every i , and $\sum_i x_i = 1$

- **k-item unit-demand auctions**

- k identical items for sale
- $x_i \in \{0, 1\}$, $\sum_i x_i \leq k$

- **Knapsack auctions**

- k identical items for sale
- For each bidder, demand of w_i units
- $x_i \in \{0, 1\}$ for every i , $\sum_i w_i x_i \leq k$

- **Public project mechanisms**

- Deciding whether to build a public project (e.g., a park)
- Only 2 feasible allocations: $(0, 0, \dots, 0)$ or $(1, 1, \dots, 1)$

Allocation rules and truthful mechanisms

- Can we understand how to derive truthful mechanisms?
- Actually, we can rephrase this as:
 - Suppose we are given an allocation rule \mathbf{x}
 - Can we tell if \mathbf{x} can be combined with a pricing rule \mathbf{p} , so that (\mathbf{x}, \mathbf{p}) is a truthful mechanism?
- This would allow us to focus only on designing the allocation algorithm appropriately
- Consider the single-item auction
 - Allocation rule 1: Give the item to the highest bidder
 - Allocation rule 2: Give the item to the 2nd highest bidder
- For rule 1, we have seen how to turn it into a truthful mechanism (Vickrey auction)
- For rule 2?
 - We have not seen how to do this, but we have also not proved that it cannot be done

Allocation rules and truthful mechanisms

- Consider a mechanism with allocation rule \mathbf{x}
- Fix a player i , and fix a profile \mathbf{b}_{-i} for the other players
- Allocation to player i at a profile $\mathbf{b} = (z, \mathbf{b}_{-i})$ is given by $x_i(\mathbf{b})$
- Keeping \mathbf{b}_{-i} fixed, we can view the allocation to player i as a function of his bid
 - $x_i = x_i(z, \mathbf{b}_{-i})$, if bidder i bids z
- Definition: An allocation rule is **monotone** if for every bidder i , and every profile \mathbf{b}_{-i} , the allocation $x_i(z, \mathbf{b}_{-i})$ to i is non-decreasing in z
- I.e., bidding higher can only get you more stuff

Monotonicity of allocation rules

Examples

- Back to the single-item auction
- The allocation rule that gives the item to the highest bidder is monotone
 - If a bidder wins at profile \mathbf{b} , she continues to be a winner if she raises her own bid (keeping \mathbf{b}_{-i} fixed)
 - If she was not a winner at \mathbf{b} , then by raising her bid, she will either remain a non-winner or she will become a winner
- The allocation rule that gives the item to the 2nd highest bidder is not monotone
 - If I am a winner and raise my bid, I may become the highest bidder and will stop being a winner

Myerson's lemma

[Myerson '81]

- **Theorem:** For every single-parameter environment,
 - An allocation rule \mathbf{x} can be turned into a truthful mechanism if and only if it is monotone
 - If \mathbf{x} is monotone, then there is a unique payment rule \mathbf{p} , so that (\mathbf{x}, \mathbf{p}) is a truthful mechanism
 - Subject to the constraint that if $b_i = 0$, then $p_i = 0$
- One of the classic results in mechanism design
- In fact, in many cases we can also compute the payments by a simple formula

Myerson's lemma

- Allocation rule \mathbf{x} is truthful \Rightarrow

Allocation rule \mathbf{x} is monotone: for all z, y , $(\mathbf{x}(z) - \mathbf{x}(y))(z - y) \geq 0$

If z is the true value:

$$\mathbf{x}(z) \cdot z - \mathbf{p}(z) \geq \mathbf{x}(y) \cdot z - \mathbf{p}(y) \quad (1)$$

If y is the true value:

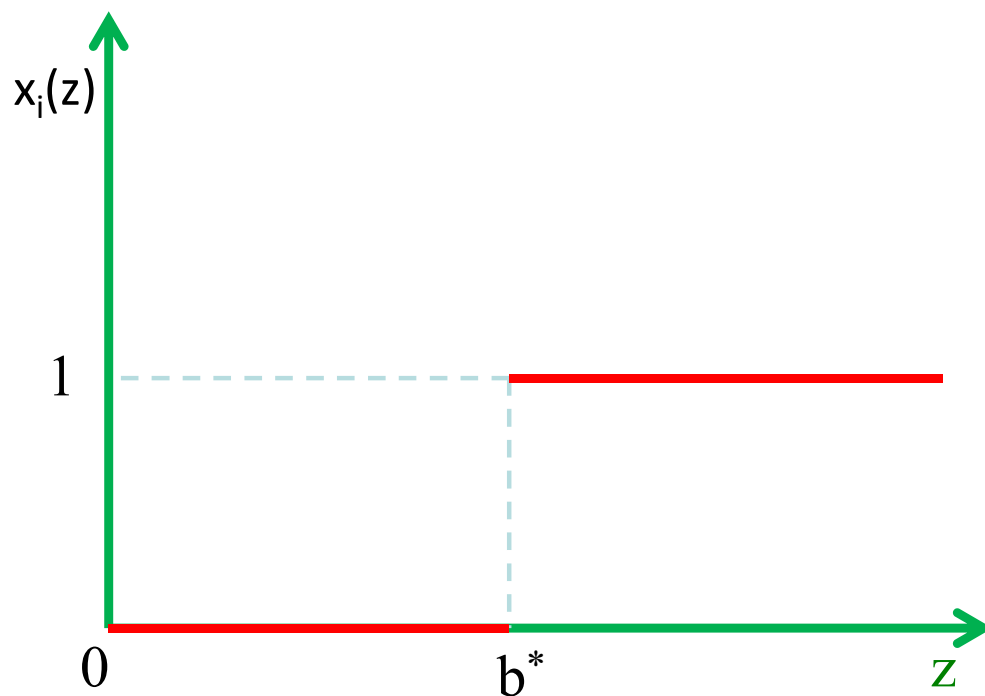
$$\mathbf{x}(y) \cdot y - \mathbf{p}(y) \geq \mathbf{x}(z) \cdot y - \mathbf{p}(z) \quad (2)$$

Summing up (1) and (2):

$$\begin{aligned} \mathbf{x}(z) \cdot z + \mathbf{x}(y) \cdot y &\geq \mathbf{x}(y) \cdot z + \mathbf{x}(z) \cdot y \Leftrightarrow \\ (\mathbf{x}(z) - \mathbf{x}(y)) \cdot z &\geq (\mathbf{x}(z) - \mathbf{x}(y)) \cdot y \Leftrightarrow \\ (\mathbf{x}(z) - \mathbf{x}(y)) \cdot (z - y) &\geq 0 \end{aligned}$$

Myerson's lemma and payment formula

- For the payment rule, we need to look for each bidder at the allocation function $x_i(z, \mathbf{b}_{-i})$
- For the single-item truthful auction:
 - Fix \mathbf{b}_{-i} and let $b^* = \max_{j \neq i} b_j$



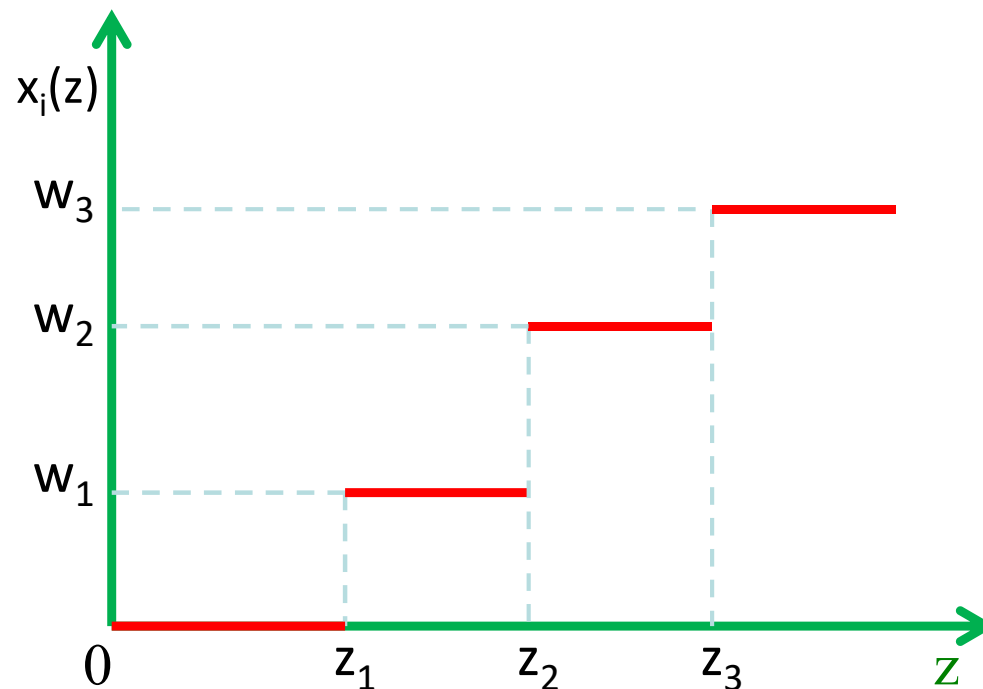
Facts:

- For any fixed \mathbf{b}_{-i} , the allocation function is piecewise linear with 1 jump
- The Vickrey payment is precisely the value at which the jump happens
- The jump changes the allocation from 0 to 1 unit

Myerson's lemma and payment formula

For most scenarios of interest

- The allocation is piecewise linear with multiple jumps
- The jump determines how many extra units the bidder wins

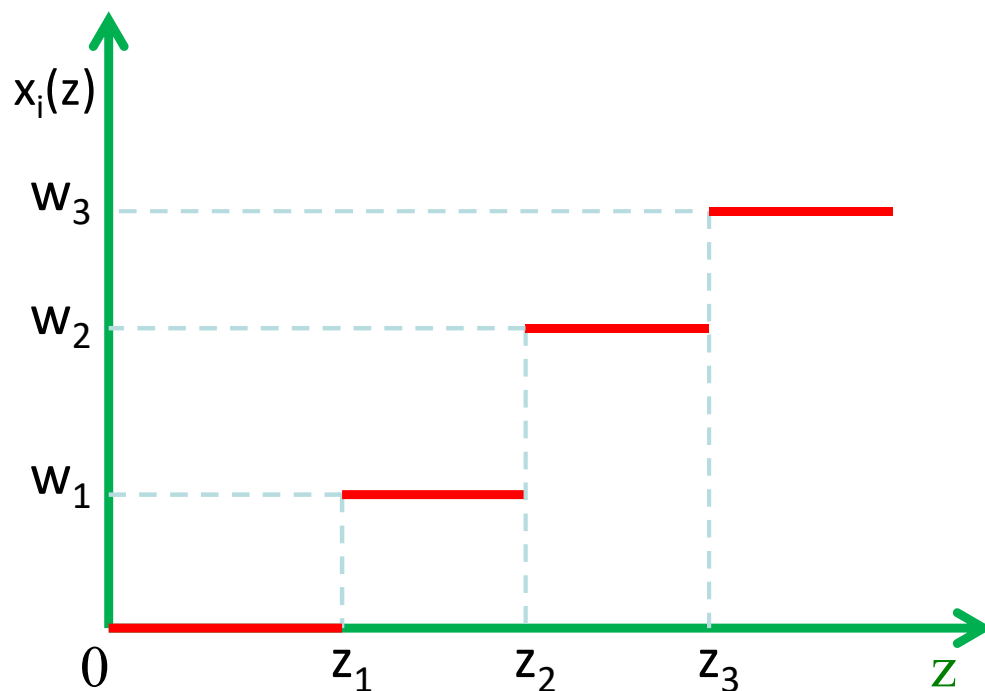


- Suppose bidder i bids b_i
- Look at the jumps of $x_i(z, b_i)$ in the interval $[0, b_i]$
- Suppose we have k jumps
- Jump at z_1 : w_1
- Jump at z_2 : $w_2 - w_1$
- Jump at z_3 : $w_3 - w_2$
- ...
- Jump at z_k : $w_k - w_{k-1}$

Myerson's lemma and payment formula

For most scenarios of interest

- The allocation is piecewise linear with multiple jumps
- The jump determines how many extra units the bidder wins



Payment formula

- For each bidder i at a profile b , find all the jump points within $[0, b_i]$
- $p_i(b) = \sum_j z_j \cdot [\text{jump at } z_j]$
 $= \sum_j z_j \cdot [w_j - w_{j-1}]$
- The formula can also be generalized for monotone but not piecewise linear functions

Myerson's lemma

- Allocation rule \mathbf{x} is truthful (and thus, monotone) => find appropriate payments \mathbf{p}

If z is the true value:

$$\mathbf{x}(z) \cdot z - \mathbf{p}(z) \geq \mathbf{x}(y) \cdot z - \mathbf{p}(y) \quad (1)$$

If y is the true value:

$$\mathbf{x}(y) \cdot y - \mathbf{p}(y) \geq \mathbf{x}(z) \cdot y - \mathbf{p}(z) \quad (2)$$

Combining (1) and (2), we get:

$$z(\mathbf{x}(z) - \mathbf{x}(y)) \leq \mathbf{p}(y) - \mathbf{p}(z) \leq y(\mathbf{x}(z) - \mathbf{x}(y))$$

Assuming that y tends to z from above, in the limit, we get:

$$\mathbf{p}'(z) = z \cdot \mathbf{x}'(z) \quad (3)$$

Myerson's lemma

- Allocation rule \mathbf{x} is truthful (and thus, monotone) \Rightarrow find appropriate payments \mathbf{p}

$$\mathbf{p}'(z) = z \cdot \mathbf{x}'(z) \quad (3)$$

We assume $\mathbf{p}(0) = 0$ (normalization) and solve (3):

$$p_i(b_i, \mathbf{b}_{-i}) = \int_0^{b_i} z \cdot \mathbf{x}'_i(z, \mathbf{b}_{-i}) dz = b_i \cdot \mathbf{x}_i(b_i, \mathbf{b}_{-i}) - \int_0^{b_i} \mathbf{x}_i(z, \mathbf{b}_{-i}) dz$$

$$p_i(b_i, \mathbf{b}_{-i}) = b_i \cdot \mathbf{x}_i(b_i, \mathbf{b}_{-i}) - \int_0^{b_i} \mathbf{x}_i(z, \mathbf{b}_{-i}) dz$$

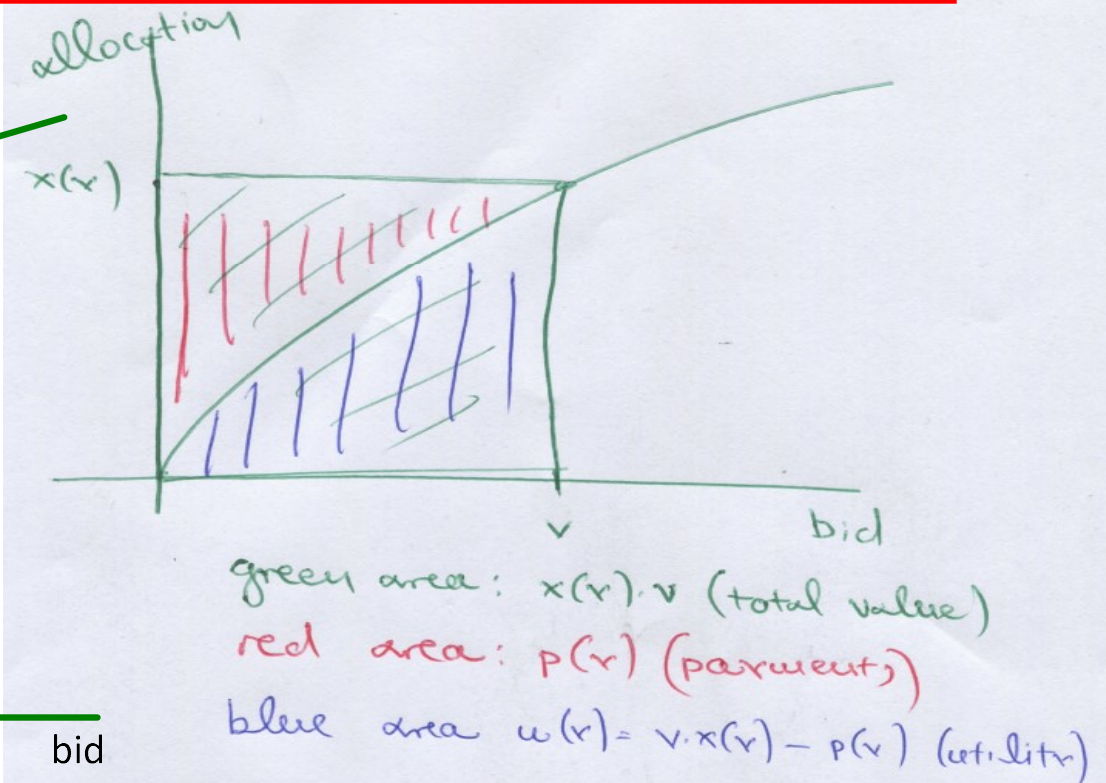
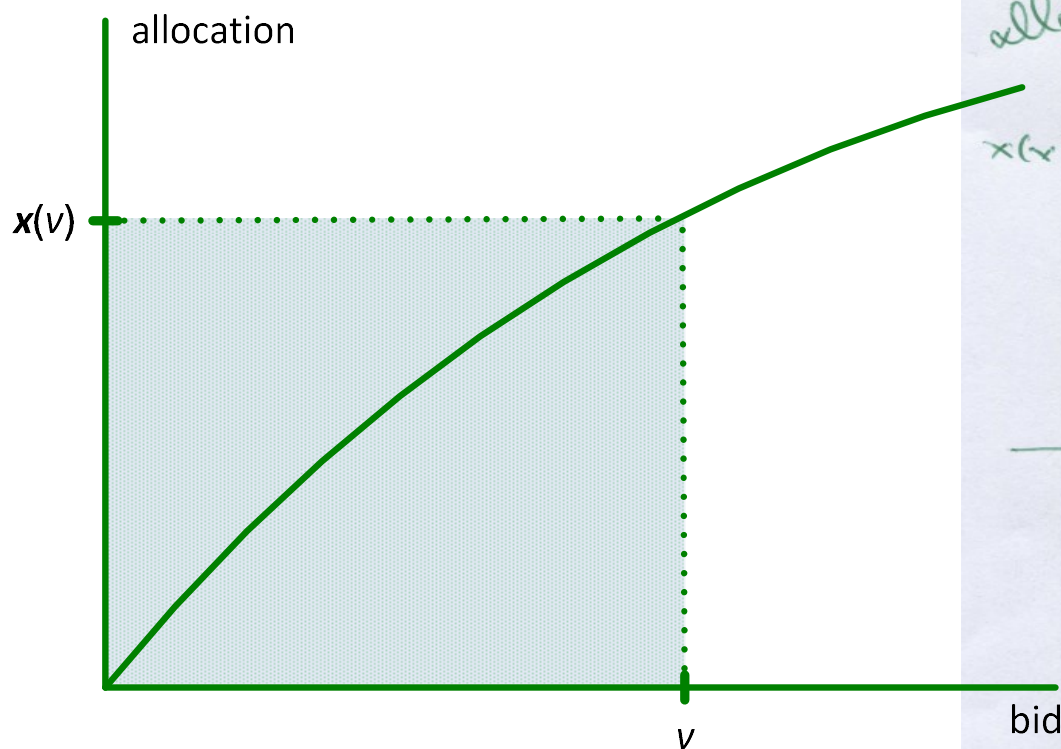
$$i\text{'s utility: } u_i(b_i, \mathbf{b}_{-i}) = (v_i - b_i) \cdot \mathbf{x}_i(b_i, \mathbf{b}_{-i}) + \int_0^{b_i} \mathbf{x}_i(z, \mathbf{b}_{-i}) dz$$

Myerson's lemma

- Any monotone allocation rule \mathbf{x} is truthful with payments \mathbf{p}

$$p_i(b_i, \mathbf{b}_{-i}) = b_i \cdot x_i(b_i, \mathbf{b}_{-i}) - \int_0^{b_i} x_i(z, \mathbf{b}_{-i}) dz$$

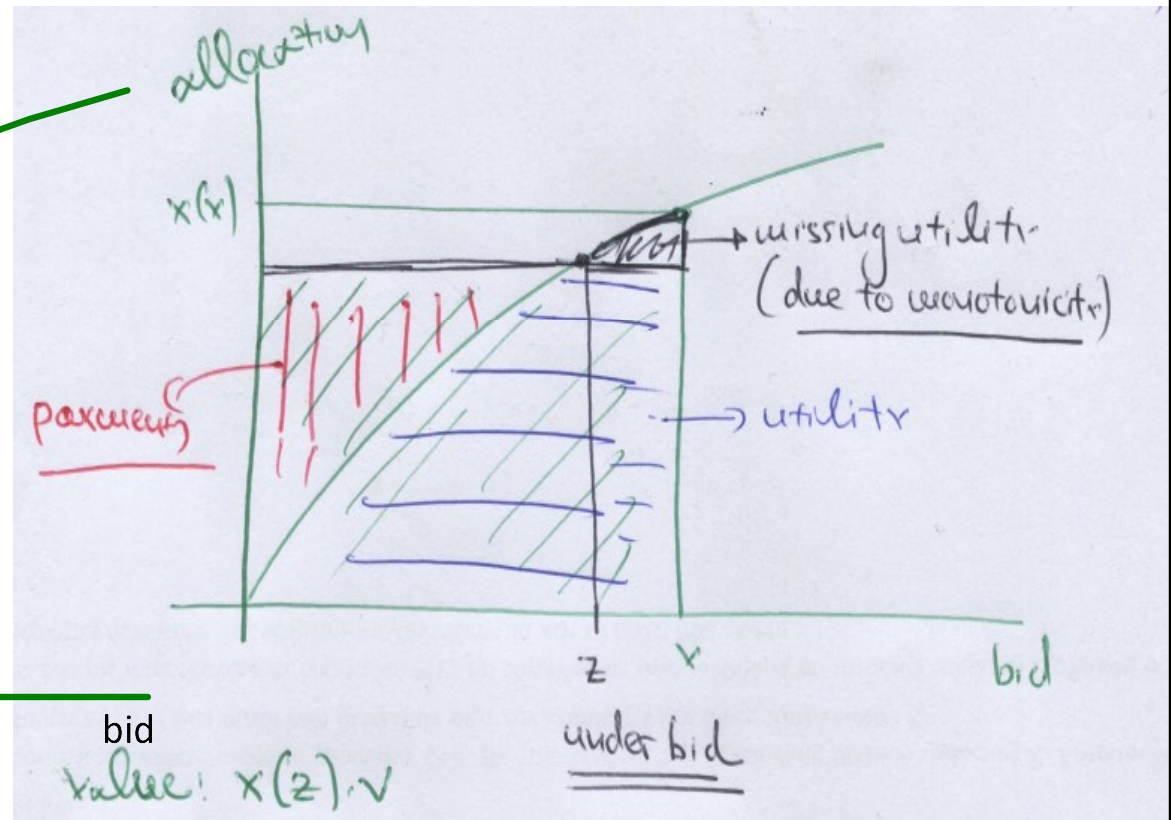
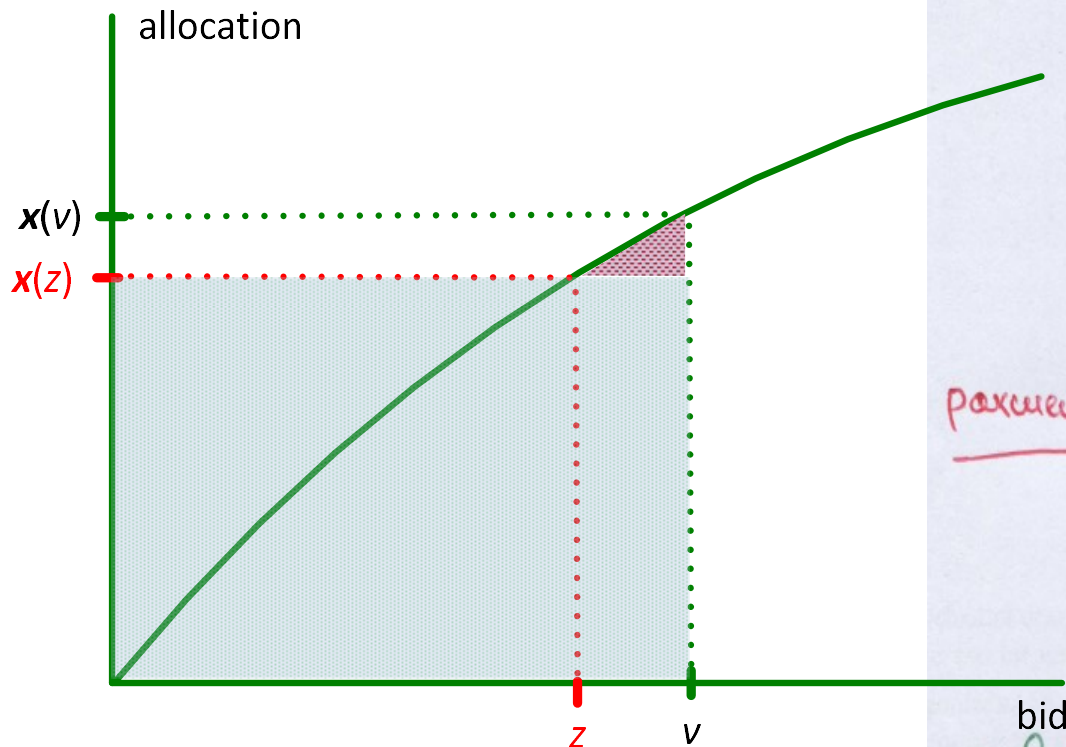
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Myerson's lemma

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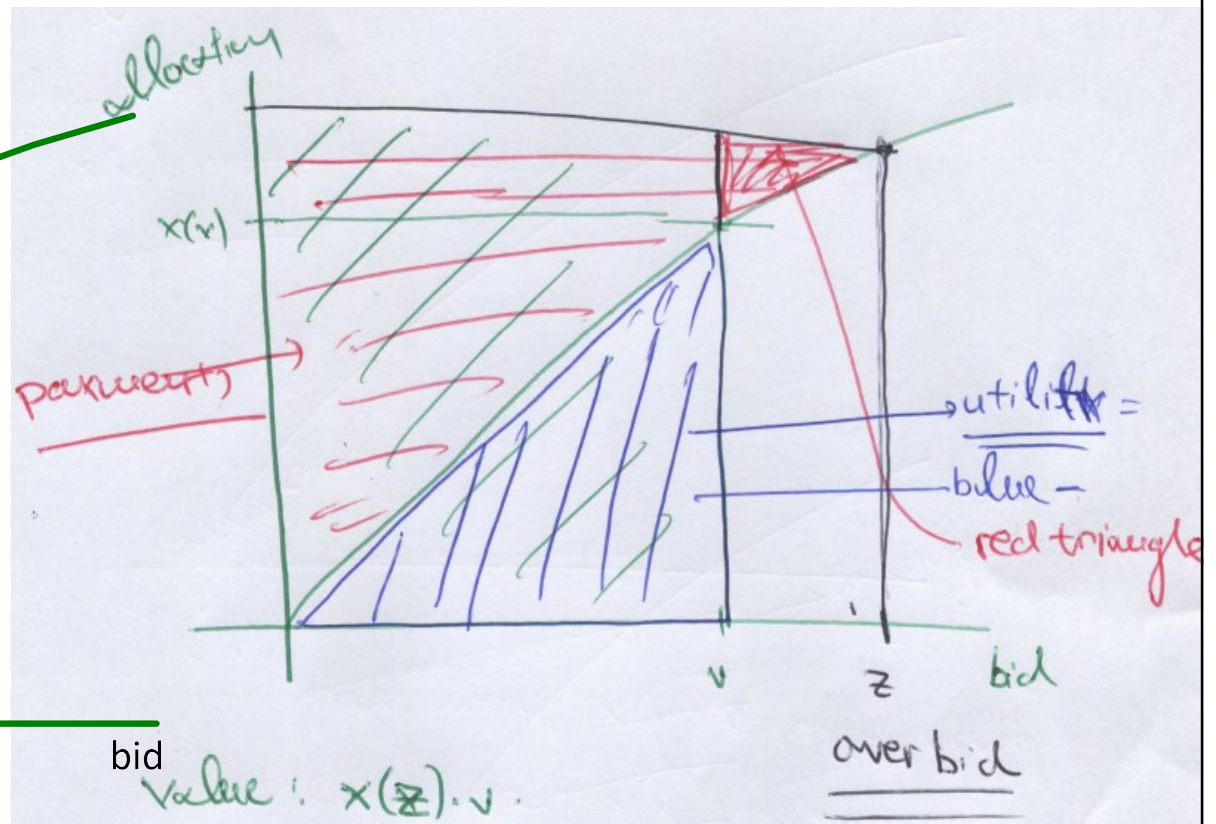
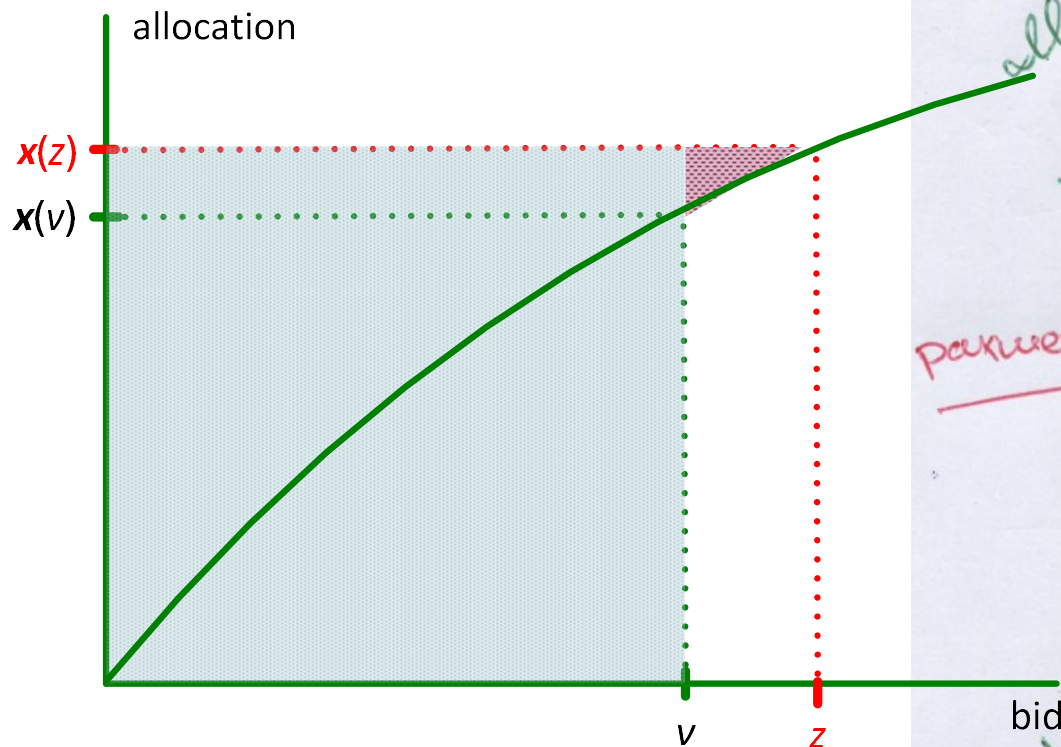
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Myerson's lemma

- Any monotone allocation rule \mathbf{x} is truthful with payments \mathbf{p}

$$i\text{'s utility: } u_i(b_i, \mathbf{b}_{-i}) = (v_i - b_i) \cdot x_i(b_i, \mathbf{b}_{-i}) + \int_0^{b_i} x_i(z, \mathbf{b}_{-i}) dz$$



Applying Myerson's lemma

- Single-item auctions
- The allocation rule of giving the item to the highest bidder is monotone
- The payment rule of Myerson gives us precisely the Vickrey auction
 - Non-winners pay nothing: If a bidder i is not a winner, there is no jump within $[0, b_i]$ in the function $x_i(z, \mathbf{b}_{-i})$
 - The winner pays $(2^{\text{nd}} \text{ highest bid}) \cdot [\text{jump at } 2^{\text{nd}} \text{ highest bid}] = 2^{\text{nd}} \text{ highest bid}$
- **Corollary:** The Vickrey auction is the only truthful mechanism for single-item auctions, when the winner is the highest bidder