THE JOURNAL OF SYMBOLIC LOGIC
Volume 53, Number 3, Sept. 1988

BELIEVING THE AXIOMS. II

PENELOPE MADDY

This is a continuation of *Believing the axioms*. I,[1] in which nondemonstrative arguments for and against the axioms of ZFC, the continuum hypothesis, small large cardinals and measurable cardinals were discussed. I turn now to determinacy hypotheses and large large cardinals, and conclude with some philosophical remarks.

§V. Determinacy. Determinacy is a property of sets of reals.[2] If A is such a set, we imagine an infinite game $G(A)$ between two players I and II. The players take turns choosing natural numbers. In the end, they have generated a real number $\mathbf{r}$ (actually a member of the Baire space ${}^{\omega}\omega$). If $\mathbf{r}$ is in A, I wins; otherwise, II wins. The set A is said to be determined if one player or the other has a winning strategy (that is, a function from finite sequences of natural numbers to natural numbers that guarantees the player a win if he uses it to decide his moves).

Determinacy is a "regularity" property (see Martin [1977, p. 807]), a property of well-behaved sets, that implies the more familiar regularity properties like Lebesgue measurability, the Baire property (see Mycielski [1964] and [1966], and Mycielski and Swierczkowski [1964]), and the perfect subset property (Davis [1964]). Infinitary games were first considered by the Polish descriptive set theorists Mazur and Banach in the mid-30s; Gale and Stewart [1953] introduced them into the literature, proving that open sets are determined and that the axiom of choice can be used to construct an undetermined set.

Gale and Stewart also raised the question of whether or not all Borel sets are determined, but the answer was long in coming. Wolfe [1955] quickly established the determinacy of Σ^0_2 games, but it was not until [1964] that Davis showed the same for Σ^0_3 games. It was [1972] before Paris was able to extend the result to Σ^0_4, and by that point the proof had become fiendishly complex. Martin then capped the whole enterprise with his surprising proof of Borel Determinacy in [1975]. (This result was

Received March 4, 1987.

[1] Maddy [BAI].

[2] My [1984] contains an earlier version of some of the material in this section. (The support of the NEH during that work is again gratefully acknowledged.) The standard (and excellent) reference for both historical and mathematical information on determinacy is Moschovakis [1980]. Notice that by "reals" modern descriptive set theorists mean members of the Baire space ${}^{\omega}\omega$, which is homeomorphic to the irrationals.

0022-4812/88/5303-0005/$03.90

mentioned in [BAI, §§I.8 and III] because of its essential use of Replacement.) Given that $V = L$ implies the existence of a nonmeasurable Δ^1_2 set, this result is the best possible; any further determinacy goes beyond ZFC.[3]

The first determinacy hypothesis, suggested by Steinhaus in Mycielski and Steinhaus [1962], was the full Axiom of Determinacy (AD), that is, the assumption that every set of reals is determined. Given that it contradicts the Axiom of Choice, the authors did not propose AD as a truth about V, but rather as applying to some substructure thereof:

> It is not the purpose of this paper to depreciate the classical mathematics with its fundamental 'absolute' intuitions on the universum of sets (to which belongs the axiom of choice), but only to propose another theory which seems very interesting ... Our axiom can be considered as a restriction of the classical notion of a set leading to a smaller universum, say of determined sets, which reflect some physical intuitions which are not fulfilled by the classical sets ([the various pathologies implied by Choice] are eliminated). Our axiom could be considered as an axiom added to the classical set theory claiming the existence of a class of sets satisfying [AD] and the classical axioms (without the axiom of choice). (p. 2)

Though the Axiom of Choice implies the existence of various extremely complex sets (for example, non-Lebesgue measurable sets, uncountable sets without perfect subsets, well-orderings of the reals, etc.), the Axiom of Determinacy might still hold is some inner model of ZF (ZFC, without Choice). This inner model would then consist only of regular sets; the irregular sets would appear in the more remote parts of V:

> We can only hope that some submodels of the natural models of [ZFC] are models of [ZFC + AD] ... It would be still more pleasant if such a submodel contains all the real numbers. In that case [AD] might be considered as a limitation of the notion of a set excluding some 'pathological' [ZFC] sets.

(Mycielski [1964, p. 205]) The smallest such model is $L[R]$, and the Axiom of Quasi-Projective Determinacy (QPD)[4] is the assumption that all sets of reals in this submodel are determined. This is the live axiom candidate (see e.g. Moschovakis [1970, p. 31] and [1980, pp. 422 and 605]; Martin [PSCN, p. 8]).[5]

---

[3]I mean, full Σ^1_1 or Π^1_1 determinacy go beyond ZFC. Modest gains beyond Borel determinacy are possible without additional assumptions. See, for example, Wolf [1985].

[4]This assumption is usually written symbolically as $AD^{L[R]}$ and otherwise unnamed. In [1969], Solovay uses the term "quasiprojective" for the sets of reals in $L[R]$, so I have adopted his terminology.

[5]A weaker assumption, the Axiom of Projective Determinacy (PD), is also discussed in the literature. (PD is naturally the assumption that all projective sets of reals are determined; it is weaker than QPD because all projective sets appear in $L[R]$.) QPD is the better axiom candidate because the projective hierarchy is only the second of a series of hierarchies, while $L[R]$ is a transitive model of ZFC generated in a natural way.

It is worth noting that QPD has a form reminiscent of a number of mathematical implications (see Fenstad [1971, p. 42]; Addison and Moschovakis [1968, p. 710]). The quantifier switch

$$\exists x \forall y\, Rxy \supset \forall y \exists x\, Rxy$$

is a theorem of logic, but the other direction

$$\forall x \exists y\, Rxy \supset \exists y \forall x\, Rxy$$

is nontrivial. Determinacy assumptions have the second form: if for every strategy for I, there is a way for II to play that results in a win for II, then there is a strategy for II that results in a win for II no matter what I plays. (In other words, if I has no winning strategy, then II does.) This same nontrivial quantifier switch is seen in various mathematical contexts, for example, in the implication from continuity to uniform continuity. An implication of this sort usually requires a simplifying assumption—in the continuity example, that the space in question is compact. In the case of QPD, the simplifying assumption is that the set in question is constructible from the reals. So QPD at least has a general form that is familiar from other parts of mathematics.

Still, as far as intrinsic evidence for QPD is concerned, even its staunchest supporters are emphatic in their denials:

> No one claims direct intuitions... either for or against determinacy hypotheses.
>
> There is no *a priori* evidence for [Q]PD.
>
> Is [Q]PD true? It is certainly not self-evident.

(Moschovakis [1980, p. 610]; Martin [1976, p. 90]; Martin [1977, p. 813]; see also Wang [1974, p. 554]). What sets QPD apart (or what did set it apart before the recent discoveries discussed in the next section) is that its defense has been purely extrinsic. Even the most skeptical among the supporters of large cardinals admit that extending the sequences of ordinals is intrinsic to the iterative conception of set (for example, *maximize*). Nothing of this type whatsoever was offered for QPD from its origin until the mid-80s.[6] Yet it has been taken very seriously as an axiom candidate.

[6]To be accurate, I should admit one exception:

> The reader who knows the Zermelo-von Neumann theorem on the strict determinateness of finite positional games could accept perhaps the following 'intuitive justification' of [AD]. Suppose that both players I and II are infinitely clever and that they know perfectly well what [the set of reals] P is, then owing to the complete information during every play, the result of the play cannot depend on chance. [AD] expresses exactly this.

(Mycielski and Steinhaus [1962, p. 1]) I ignore this argument for two reasons. First, it supports, if anything, the full, false, AD. Second, it has not been adopted by subsequent researchers.

On the other hand, the appeal to extrinsic supports has been quite explicit:

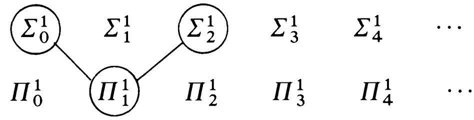
> The author regards [Q]PD as an hypothesis with a status similar to that of a theoretical hypothesis in physics ... quasi-empirical evidence for [Q]PD has been produced.
>
> ... those who have come to favor these hypotheses as plausible, argue from their consequences ... the richness and internal harmony of these consequences.
>
> In the case of [Q]PD the evidence is mostly *a posteriori*: its consequences look right.
>
> There is a good deal of *a posteriori* evidence for it.

(Martin [1977, p. 814]; Moschovakis [1980, p. 610]; Martin [PSCN, p. 8]; Martin [1976, p. 90]). I will sketch the three main types of extrinsic evidence—from consequences, from intertheoretic connections, and from the "naturalness" of game theoretic proofs—in the next three subsections. In the final subsection, I will consider the relevance of determinacy assumptions to the continuum problem.

V.1. *Welcome consequences.* Recall that the existence of a Δ^1_2 well-ordering of the reals, a Δ^1_2 non-Lebesgue measurable set, and an uncountable Π^1_1 set with no perfect subset were all counted as extrinsic disconfirmation for $V = L$ (see BAI, §§ II.2 and II.3.1]). It was felt that these Choice-generated oddities should not appear among the simpler sets, that they should probably not be definable at all. This might be counted as a rule of thumb in favor of the *banishment* of such sets to remote regions of V far beyond the simple sets. Of course, QPD does exactly this; it contradicts $V = L$ by forcing these "irregular" sets of reals out of the projective hierarchy, and indeed, out of $L[R]$. Thus these consequences of $V = L$ are called "a defect in that theory" (Fenstad [1971, p. 59]) or "unpleasant consequences of that theory" (Martin [1977, p. 806]), while the corresponding consequences of QPD are "pleasing consequences for the behavior of projective sets" (Martin [1976, p. 90]; see also Martin [1977, p. 811]).

Another set of welcome consequences concerns the "structural" properties of the projective sets: separation, reduction and uniformization. In the 30s, Kuratowski, Lusin and Novikov established reduction and uniformization for Σ^1_0, Π^1_1, and Σ^1_2, and separation for their opposites, Π^1_0, Σ^1_1, and Π^1_2.[7] Thus the reduction and uniformization principles hold for the circled classes, while separation holds on the opposite side:

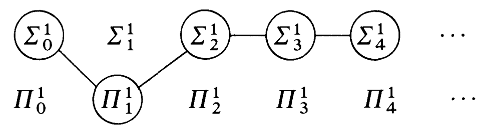
ZFC



[7] For definitions, proofs and references, see Moschovakis [1980, pp. 33–35 and 4B.10, 4B.11, and 4D.4]. Here Σ^1_0 is taken to be the open sets of reals and Π^1_0 the closed. More on this choice of notation below.

Nothing more was known until [1959], when Addison used the Δ^1_2 well-ordering of the reals in L to show that there reduction and uniformization continue on the Σ side, and separation on the Π side, for the remainder of the projective hierarchy:

$V = L$



There the matter stood for nearly ten years.

The question of what pattern of structural properties to expect at the Σ^1_3/Π^1_3 level and beyond was somewhat overshadowed in the early 60s by the new forcing industry. Among those interested in the truth of this matter, rather than in relative consistency results, opinion varied. Some felt that the use of a measurable cardinal at the next level should push through the same pattern as $V = L$. Speaking in 1967, Addison considers this possibility:

> This [Σ-side] pattern at level 3 follows from the existence of a Δ^1_2 well-ordering of [the reals], which follows in turn from the axiom of constructibility. One possibility is that higher "axioms of infinity" such as the axiom of measurable cardinals might imply this pattern at the third level. From results of Silver it is known that this matter at the third level is at least consistent with the axiom of measurable cardinals. On the other hand the axiom of measurable cardinals wipes out some nice well-orderings of [the reals] and it is thought by some that still higher axioms of infinity may be found which wipe out all projective well-orderings of [the reals]. Although nice well-orderings can be viewed as pushing in the direction of [the Σ-side] pattern, weaker principles not ruled out by higher axioms of infinity might still be enough to force it.

(Addison [1974, p. 9]) Others expected the pattern to continue alternating. Addison again:

> On the other hand if there is indeed some pressure, not yet understood, pushing for the separation principle to hold on one side or the other then it might be sufficient... to push through a[n alternating] pattern at level 3. This might look surprising, but at least one respected logician has suggested it. It has the advantage of prolonging the alternation... (p. 10)

It should be noted that the second group outnumbered the first, and that it included Gödel (Addison [1974, p. 10]).

What reasons could be given for or against the alternating pattern? The structural properties at level three and above were strongly suspected of being independent, although this was not proved until much later (see Moschovakis [1980, p. 284]). Those who expected the continuation of the Σ-side pattern of $V = L$ had a powerful new hypothesis to work with (MC), one that had only recently begun producing results about projective sets (Solovay [1969]). Silver's work on $L[U]$ showed that their conjecture was relatively consistent, and the similarity of that model to L made them expect the same pattern. Meanwhile, those favoring the alternating picture were without a new assumption, but they were supported by the brute fact that almost any human being will judge  to be a "more natural" continuation of  than . (This fact is slightly compromised by its dependence on the identification of Σ_0^1 with Σ_1^0. Without this, the first "zig" of the "zigzag" is lost.[8] But see the next quotation from Addison below.)

Moschovakis had a deeper reason for expecting the alternation to continue. In the mid-60s, he showed how the prewellordering property could be used to lift the structural theory of Π_1^1 sets to Σ_2^1. A prewellordering misses being a full wellordering by lacking antisymmetry; equivalently, it is a mapping onto an ordinal. A class of sets of reals has the prewellordering property (PWO) if every set in it admits a prewellordering that meets a delicate definability condition (see Moschovakis [1980, 4B] for details). PWO(Π_1^1) is essentially a classical theorem proved by Lusin and Sierpiński in 1923 using something called the Lusin-Sierpiński ordering.[9]

Since the prewellordering property is the key to the structural properties of the projective classes, Moschovakis's idea was to prove:

$$\mathrm{PWO}(\Pi_1^1) \Rightarrow \mathrm{PWO}(\Sigma_2^1)$$

thus effectively lifting the theory of Π_1^1 to Σ_2^1. The proof takes a simple form. Suppose A is Σ_2^1. Then there is a Π_1^1 B such that

$$A = \{x \mid \exists z((x, z) \in B\}.$$

If f maps B onto an ordinal as PWO(Π_1^1) requires, then a suitably definable prewellordering of A is achieved by taking infimums: for x, y in A,

$$x \lesssim y \quad \text{iff} \quad \inf\{f(z) \mid (x, z) \in B\} \leq \inf\{f(z) \mid (y, z) \in B\}.$$

In fact, this proof is perfectly general; whenever PWO holds at Π_n^1 it can be lifted to Σ_{n+1}^1.

If a proof using infimums moves the prewellordering property from Π_n^1 to Σ_{n+1}^1, shouldn't a proof using supremums move it from Σ_n^1 to Π_{n+1}^1? Of course this form of proof cannot work without a new hypothesis, because Addison's results from $V = L$ show that PWO(Σ_3^1) and not-PWO(Π_3^1) are relatively consistent. The trouble is that the prewellordering defined using supremums is often trivial, so the definability condition does not hold unless the set in question is more special than Π_3^1. Still, Moschovakis felt the failed argument was a "false, but natural" proof, too reasonable to be completely off-base, too natural to be a totally wrong idea. The flaw seemed akin to dividing by zero in a proof that is otherwise in order; some minor

[8] Opinion on this matter varies. Some would find it more natural to start the projective hierarchy from the Borel sets, which would destory the first leg of the alternation. It is worth noting that the Σ_1^0 beginning does not work for the actual reals (see Martin [1977, p. 790]). (Recall that modern descriptive set theory is done on the Baire space ${}^{\omega}\omega$ instead.)

[9] The "lightface" or effective version of the same theorem was proved in 1955 by Kleene. There the ordering is called the Brouwer-Kleene ordering.

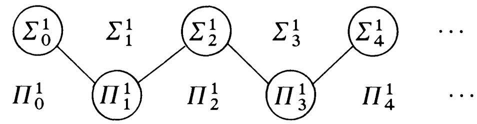
adjustment—multiplying through by a factor before dividing, or some such thing—should be enough to make it go through. This line of thought led Moschovakis to the conjecture that PWO(Π^1_3), and to favor the alternating pattern.[10]

Then came Blackwell [1967], in which an elegant new proof of Lusin's classical theorem on the separability of Σ^1_1 sets is derived from the determinacy of open sets. Addison and Martin quickly adopted the method to show that the reduction property for Π^1_3 sets could be derived directly from the determinacy of Δ^1_2 sets. When Moschovakis heard of this work, he had his additional hypothesis. The "false but natural" proof could be revived by requiring one supremum to be "effectively" smaller than the other, where "effectively" is parsed out in terms of a determined game. In general, then, we get the periodicity theorem:

$$\mathrm{Det}(\Delta^1_n) \Rightarrow (\mathrm{PWO}(\Sigma^1_n) \Rightarrow \mathrm{PWO}(\Pi^1_{n+1})).$$

(See Addison and Moschovakis [1968]. Martin proved the same thing independently, using degree theoretic methods, in [1968].) Thus under the assumption of QPD, the alternating pattern of structural properties continues for the remainder of the projective hierarchy:

QPD



This result—the extension of the alternating pattern—is now considered strong extrinsic evidence in favor of QPD. In a footnote added to the pro-alternation paragraph in the printed version of his talk, Addison remarks:

> This paragraph turned out to be prophetic. Only a month or so after the talk was given it was shown that [QPD] does indeed give the alternation of ... patterns discussed here ... Moreover the reasons mentioned above for the plausibility of this hypothesis actually lie behind the proof of the alternating pattern from ... determinateness. Furthermore the idea of considering Σ^0_1 and Π^0_1 as the first level of the projective hierarchy is not only completely vindicated by the outcome but indeed actually suggested the structure of the proof of the alternating pattern. [1974, p. 10]

(This last because Blackwell's proof depended on the determinacy of Δ^0_1 sets.) Nowadays we read:

> Which is the correct picture is perhaps not absolutely clear yet, but it is fair to say that many people working in this area and prone to speak of truth in set theory (ourselves included) tend to favor the alternating picture.
>
> ... there is something odd in the sequence Σ^1_0, Π^1_1, Σ^1_2, Σ^1_3, Σ^1_4, ..., the sequence Σ^1_0, Π^1_1, Σ^1_2, Π^1_3, Σ^1_4, ... seems more plausible.

[10] Notice that it was the structure of a proof (or an attempted proof) that produced this conjecture. Philosophers are often tempted to think that conjectures are formed by a process quite independent of proof, but this case suggests otherwise.

(Moschovakis [1980, pp. 33–34]; Martin [1977, pp. 806 and 811]; see also Martin [PSCN, p. 8]; Fenstad [1971, p. 59]; Wang [1974, pp. 547 and 553–554]).

To summarize these extrinsic supports:

> [Q]PD has pleasing consequences about the behavior of projective sets, such as: Every projective set is Lebesgue measurable; Every uncountable projective set has a perfect subset. More impressive is the fact that [Q]PD allows one to extend the classical structural theory of projective sets, which dealt only with the first two levels of the projective hierarchy, to a very elegant and essentially complete theory of the projective sets. [Q]PD cannot be proved in ZFC... but it is not unreasonable to suspect that it may be true.

(Martin [1976, p. 90]) The full extent of this beautiful and remarkably detailed theory of the projective sets is clearly laid out in Moschovakis's [1980].[11]

V.2. *Intertheoretic connections.* Despite the early (erroneous) suggestion that the Axiom of Measurable Cardinals (MC) might force the "wrong" resolution for the structural properties at the third level, measurable cardinals and determinacy hypotheses were soon found to point in the same direction. Often QPD will extend a result provable from MC which in turn extends a result provable in ZFC. To use an example that has already been discussed, ZFC implies that every uncountable Σ^1_1 set has a perfect subset, MC implies every uncountable Σ^1_2 set has a perfect subset, and QPD implies every uncountable quasiprojective set has a perfect subset. Another case in point:

$$\mathrm{ZF} \Rightarrow \text{Every } \Sigma^1_2 \text{ set is the union of } \aleph_1 \text{ Borel sets,}$$
$$\mathrm{MC} \Rightarrow \text{Every } \Sigma^1_3 \text{ set is the union of } \aleph_2 \text{ Borel sets,}$$
$$\mathrm{QPD} \Rightarrow \text{Every } \Sigma^1_4 \text{ set is the union of } \aleph_3 \text{ Borel sets.}$$

(Sierpinski [1925]; Martin [PSCN]) In other cases, the same result can be proved using either MC or a determinacy hypothesis:

> To take one example, the fact that Π^1_2 sets can be uniformized by Π^1_3 sets follows both from MC and from Det(Δ^1_2), but by proofs which (at least on the surface) are totally unrelated; one tends to believe the result then and consequently to take both proofs seriously and to feel a little more sympathetic towards their respective hypotheses.

(Moschovakis [1980, p. 610]) But these observations mark only the beginning of the deep connection between measurable cardinals and lower forms of determinacy.

The first two results suggesting this connection were Solovay's of 1967:

$$\mathrm{AD} \Rightarrow \aleph_1 \text{ is a measurable cardinal}$$

[11] It has been conjectured that these implications might be reversed, that is, that a strong determinacy hypothesis might be derivable from the assumption that this rich theory holds of the projective sets. This would obviously provide considerable additional support for determinacy.

and Martin's in [1970]:

$$\mathrm{MC} \Rightarrow \mathrm{Det}(\Sigma^1_1).$$

In fact, Martin's theorem only depends on the existence of the sharps, and by [1978], Harrington had proved the converse:

$$\forall x(x^\# \text{ exists}) \equiv \mathrm{Det}(\Sigma^1_1).$$

Meanwhile, Solovay (building on work of Martin and Friedman) improved his result to:

$$\mathrm{Det}(\Delta^1_2) \Rightarrow \text{There are inner models with many MCs.}$$

Martin saw the development of his and Harrington's results as following a pattern: a large cardinal axiom (MC) implies some determinacy assumption ($\mathrm{Det}(\Sigma^1_1)$); careful analysis reveals that the hypothesis can be weakened to the existence of an inner model with a slightly smaller large cardinal and indiscernibles (here ZFC itself is viewed as "large cardinal assumption"); finally, the implication is improved to an equivalence. In view of Solovay's result, this pattern might be extended within the Δ^1_2 sets.

To get a finer breakdown of the Δ^1_2 sets, Martin turned to the "difference hierarchy" of Π^1_1 sets: A is α-Π^1_1 iff there is a sequence $(A_\beta\colon \beta < \alpha)$ such that each A_β is Π^1_1 and $x \in A$ iff the least β such that ($\beta = \alpha$ or $x \notin A_\beta$) is odd. (Limit ordinals are even.) Thus A is 1-Π^1_1 iff A is Π^1_1; A is 2-Π^1_1 iff A is a difference of Π^1_1 sets; A is 3-Π^1_1 iff A is the union of a difference of Π^1_1 sets and a Π^1_1 set; A is 4-Π^1_1 iff A is the union of two differences of Π^1_1 sets; and so on. The finite levels of this hierarchy generate all the Boolean combinations of Π^1_1 sets.

The theorem, then, is:

$$\mathrm{Det}((\omega^2 \cdot \alpha + 1)\text{-}\Pi^1_1) \equiv \forall x(\text{there is an inner model of ZFC containing } x \text{ with indiscernibles and } \alpha \text{ MCs}).$$

For $\alpha = 0$, this is exactly the Martin/Harrington equivalence. For $\alpha = 1$, it is:

$$\mathrm{Det}((\omega^2 + 1)\text{-}\Pi^1_1) \equiv \forall x(\text{there is an inner model of ZFC containing } x \text{ with indiscernibles and one MC}).$$

The canonical model $L[U]$ has one measurable cardinal and indiscernibles, and the set of formulas that codes its construction (just as $0^\#$ codes the construction of L) is called $0^\dagger$. If $x^\dagger$ is defined analogously with $x^\#$, the theorem for $\alpha = 1$ can be written

$$\mathrm{Det}((\omega^2 + 1)\text{-}\Pi^1_1) \equiv \forall x(x^\dagger \text{ exists}).$$

Thus the pattern continues: the existence of two measurable cardinals implies $\mathrm{Det}((\omega^2 + 1)\text{-}\Pi^1_1)$. Careful analysis reveals that the hypothesis can be reduced to the existence of an inner model with one measurable cardinal and indiscernibles. Finally, the implication can be reversed. The general form of Martin's theorem shows that this pattern continues through the entire difference hierarchy of Π^1_1 sets.

Thus simple game-theoretic hypotheses are equivalent to the inner model versions of measurable cardinal hypotheses for many natural classes of sets of reals within Δ^1_2.[12] This wonderful and surprising correspondence between powerful and well-supported hypotheses of such different character counts as extrinsic evidence for both.

V.3. *The naturalness of game-theoretic proofs.* Finally, there is what those involved call the "naturalness" of the proofs from QPD:

> In fact, the most persuasive argument for accepting [quasi]-projective determinacy (aside from Martin's proof of $\mathrm{Det}(\Sigma^1_1)$) is the naturalness of the known proofs of [the periodicity theorem], both Martin's and ours.

(Moschovakis [1970, p. 34]) Given our look at the development of that theorem in V.1, it is easy to see what Moschovakis has in mind here. Not only does QPD imply $\mathrm{PWO}(\Pi^1_3)$ and the rest, it does so by means of an argument that was previously thought to be of the correct sort. The proof is "natural".

Another aspect of "naturalness" is revealed when the new game-theoretic proofs yield new, simpler proofs of old theorems, and recast them as special cases of new more powerful theorems:

> One [reason for believing QPD] is the *naturalness* of proofs from determinacy—in each instance where we prove a property of Π^1_3 (say from $\mathrm{Det}(\Delta^1_2)$), the same argument gives a new proof of the same (known) property of Π^1_1, using only the determinacy of clopen sets (which is a theorem of ZF). Thus the new results appear to be natural generalizations of known results and their proofs shed new light on classical descriptive set theory. (This is not the case with the proofs from $V = L$ which all depend on the [Δ^1_2] well-ordering of [the reals] and shed no light on Π^1_1.)

(Moschovakis [1980, p. 610]) The periodicity theorem itself gives an example of this phenomenon. Recall that the classical proof of $\mathrm{PWO}(\Pi^1_1)$ involved the special properties of Π^1_1, in particular, the Lusin-Sierpiński ordering. Now for $n = 0$, the periodicity theorem is

$$\mathrm{Det}(\Delta^1_0) \Rightarrow (\mathrm{PWO}(\Sigma^1_0) \Rightarrow \mathrm{PWO}(\Pi^1_1)).$$

But determinacy of Δ^1_0 sets is just the Gale-Stewart theorem, and it is simple to show $\mathrm{PWO}(\Sigma^1_0)$. Thus the periodicity theorem provides a new proof of $\mathrm{PWO}(\Pi^1_1)$ that avoids such complexities as the Lusin-Sierpiński ordering. (See also Moschovakis [1980, p. 309].)

Another example is provided by Solovay's proofs [1969] that the regularity properties of Σ^1_1 sets could be lifted to Σ^1_2 assuming the Axiom of Measurable

---

[12] This sequence of results can be extended further. For example, Simms has shown that the existence of a measurable cardinal which is the limit of measurably many measurable cardinals implies the determinacy of countable unions of Boolean combinations of Π^1_1 sets. The hypothesis can be improved to the existence of an inner model with indiscernibles and a proper class of measurable cardinals. Then the implication can be reversed.

Of course it would be nicer if the determinacy assumptions could prove the large cardinal hypotheses outright, but this is impossible. If, for example, κ is the first measurable cardinal, the R_κ is a model of $\mathrm{Det}(\Sigma^1_1)$ but not of MC. Thus the inner model equivalences are the best possible.

Cardinals. When Martin showed that Det(Σ^1_1) could be derived from MC, he opened the way for game-theoretic proofs of these results. These new proofs avoid the complex forcing constructions of Solovay's original versions (see Moschovakis [1980, pp. 375–378, 544–546, 611]).

V.4. *Relevance to the continuum problem.* QPD gives us lots of information about the projective sets; what can it tell us about the size of the continuum? The quick answer is that it cannot settle the continuum hypothesis. (It will be easy to see why from the result of the next section.) Still, it might give us evidence for or against, or, even better, it might lead us in the direction of a larger theory that does decide the question.

Under the assumption of QPD, the perfect subset property is extended to cover the entire quasiprojective hierarchy, so the CH holds for all quasiprojective sets. As mentioned earlier [BAI, §II.3.1], the fact that CH holds for many simple sets might have been considered as evidence its favor, except that the perfect subset property is known not to hold for all sets of reals. Thus this consequence of QPD does not really provide evidence in favor of CH.

What is at issue here is the length of the shortest well-ordering of the reals. Since a definable well-ordering yields a definable non-Lebesgue measurable set, and QPD implies that all quasiprojective sets are Lebesgue measurable, it also implies that there is no quasiprojective well-ordering of the reals. This is as it should be (see V.1). In fact, the perfect subset property implies that every quasiprojective well-ordering of a set of reals is countable. This means that no projective well-ordering can provide a counterexample to CH; we cannot test the CH by looking at the projective well-orderings.

What we can do is look into the lengths of projective prewellorderings:

> Now every Σ^1_1 prewellordering has countable length, but there is a Π^1_1 prewellordering of [the reals] of length $\aleph_1$. This already shows that our simple sets are more typical with respect to prewellorderings than with respect to well-orderings.

(Martin [1976, p. 89]) In particular, consider: $\delta^1_n = \sup\{\text{length of } R \mid R \text{ is a } \Delta^1_n$ prewellordering of the reals$\}$. Information about these "projective ordinals" is information about the length of the continuum. It is a classical theorem that $\delta^1_1 = \aleph_1$; if any δ^1_n is greater than $\aleph_1$, then the continuum hypothesis is false.

The best way to approach the question of the size of the projective ordinals under QPD is to investigate them first under the full, false, AD, then to transfer the results to $L[R]$. In the strange world of full AD, it is known that the projective ordinals form a strictly increasing sequence of regular cardinals, in particular:

$$\begin{aligned} AD \Rightarrow \delta^1_1 &= \aleph_1, \\ \delta^1_2 &= \aleph_2, \\ \delta^1_3 &= \aleph_{\omega+1}, \\ \delta^1_4 &= \aleph_{\omega+2}, \\ \delta^1_5 &= \aleph_{\omega(\omega^\omega)+1}, \\ \delta^1_6 &= \aleph_{\omega(\omega^\omega)+2}. \end{aligned}$$

(These results are due to many researchers, among them Martin, Solovay, Kunen, Mansfield, Shoenfield and Jackson. See Moschovakis [1980, 7D.11]. Incidentally, the cardinals between $\aleph_2$ and $\aleph_{\omega+1}$ are all singular assuming AD; the projective ordinals are not only regular, but measurable.) This means that in the strange world of full determinacy, the continuum hypothesis is false in the sense that the reals can be mapped onto very large ordinals. In the real world, the Axiom of Choice would then yield very large subsets of the reals, but Choice does not hold in the AD world. There, remember, all uncountable sets have perfect subsets. Thus the CH is true in the sense that there are no sets of reals of intermediate cardinality, but false in the sense that the reals can be mapped onto large ordinals. As far as the actual cardinality of the reals is concerned, in the world of full AD it is not an aleph at all, because the reals cannot be well-ordered.

What does this mean for the real world, on the assumption that both QPD and Choice hold there? Since QPD is the hypothesis that AD holds in $L[R]$, the results above hold unchanged in that inner model. From this it follows that:[13]

$$\begin{aligned} \text{QPD} \Rightarrow \delta^1_1 &= \aleph_1, \\ \delta^1_2 &= (\aleph_2)^{L[R]} \le \aleph_2, \\ \delta^1_3 &= (\aleph_{\omega+1})^{L[R]} \\ &= (\text{the first regular cardinal after } \aleph_2)^{L[R]} \le \aleph_3, \\ \delta^1_4 &\le (\aleph_3^+)^{L[R]} \le \aleph_4. \end{aligned}$$

Recently, Jackson has shown that AD implies that there are exactly three regular cardinals between δ^1_3 and δ^1_5. By reasoning similar to what gave us the above, this means that

$$\text{QPD} \Rightarrow \delta^1_5 \le \aleph_7.$$

Of course it is relatively consistent that all these inequalities are strict, and that all the projective ordinals are in fact $\aleph_1$. On the other hand, $\delta^1_2 = \aleph_2$ is also relatively consistent, and for someone looking for a theory to imply the falsity of CH, QPD would seem to make a good beginning:

> ... while our simple sets have not provably given us a counterexample to CH, the possibility that they *are* counterexamples definitely arises.

(Martin [1976, p. 89]) Working with Jackson's three intermediate cardinals in the context of QPD, Martin came to conjecture that the true picture might be

[13] Here we have another example of QPD extending a pattern that begins in ZFC and continues under MC:

$$\begin{aligned} \text{ZFC} \Rightarrow \delta^1_1 &= \aleph_1, \\ \delta^1_2 &\le \aleph_2, \\ \text{MC} \Rightarrow \delta^1_3 &\le \aleph_3, \\ \text{QPD} \Rightarrow \delta^1_4 &\le \aleph_4. \end{aligned}$$

something like this:

$$\begin{array}{rcl} \text{Regular Cardinals in } L[R] & & \text{Regular Cardinals in } V \\ \aleph_1 = & \delta^1_1 & = \aleph_1 \\ \aleph_2 = & \delta^1_2 & = \aleph_2 \\ \aleph_{\omega+1} = & \delta^1_3 & = \aleph_3 \\ \aleph_{\omega+2} = & \delta^1_4 & \\ & \alpha & \\ & \beta & = \aleph_4 \\ \aleph_{\omega^{(\omega^\omega)}+1} = & \delta^1_5 & = \aleph_5 \end{array}$$

where α and β are the two other regular cardinals between δ^1_3 and δ^1_5 in the AD world of $L[R]$.

This complex conjecture can be partly confirmed by an assumption on saturated ideals developed independently by Foreman and others (for related principles, see Foreman, Magidor and Shelah [MM]). The saturated ideal hypothesis, along with QPD, implies that $\delta^1_5 \leq \aleph_5$. It remains possible that $\delta^1_n = \aleph_n$ for odd n, but a new hypothesis would be needed, presumably one that would help us understand why $L[R]$ produces so many false cardinals, both regular and singular, between the first few regular cardinals of V. Thus the best that can be said is that the rich theory of the projective ordinals provided by determinacy hypotheses might one day contribute to a theory that could falsify CH. Of course, QPD might eventually play a role in a theory that verifies CH instead, and some members of the Cabal lean toward this possibility.

§VI. Large large cardinals—down from above. By the early 70s, then, the most productive and appealing new axiom candidate, QPD, was supported exclusively by extrinsic evidence. Still, there was hope that an intrinsic connection could be found:

> Some set theorists consider large cardinal axioms self-evident, or at least as following from *a priori* principles [rules of thumb?] implied by the concept of set. Det(Σ^1_1) follows from large cardinal axioms. It is possible that [Q]PD itself follows from large cardinal axioms, but this remains unproved.
>
> One way to increase the evidence for [Q]PD would be to prove it from large cardinal axioms ...

(Martin [1977, p. 813]; Martin [PSCN, p. 8]; see also Martin [1976, p. 90]). The inner model $L[U]$ discussed in [BAI, §IV], contains a measurable cardinal and a Δ^1_3 well-ordering of the reals, so its Δ^1_3 sets are not all determined. Thus it was clear that a more powerful large cardinal would be needed.

Meanwhile, inspired by the success of measurable cardinals, and the isolation of the simpler, structural characterization in terms of elementary embeddings, Solovay and Reinhardt produced stronger large cardinal axioms. I will discuss the first of these in the next subsection, and two rule of thumb arguments for its existence in the subsection following. The very largest of the large cardinals will then be introduced, and the final subsection traces the recently-revealed connections with determinacy assumptions.

VI.1. *Supercompactness.* Recall that the ultrafilter on a measurable cardinal κ generates a nontrivial elementary embedding of V into a transitive M, and conversely, that the first ordinal moved by such an elementary embedding, the "critical point", must be measurable. Many of the strong properties of measurable cardinals spring from M's closure under arbitrary sequences of length κ, but M is not closed under longer sequences. Thus, the search for a strengthening of the Axiom of Measurable Cardinals naturally led Solovay and Reinhardt to try imposing stronger closure conditions on the range of the elementary embedding. This idea led to the notion of supercompactness:

> Then all the desired fruit, suddenly ripened, were easily plucked, and appropriately enough, the new concept was dubbed *supercompactness.*

(Kanamori and Magidor [1978, p. 183]) Specifically, a cardinal κ is λ-supercompact (for $\lambda \geq \kappa$) iff there is a nontrivial elementary embedding of V into a transitive M with κ critical and M closed under arbitrary sequences of length λ; a cardinal κ is supercompact iff κ is λ-supercompact for all $\lambda \geq \kappa$.

The connections between measurability and supercompactness are quite simple: κ is measurable iff it is κ-supercompact, and below a supercompact κ there are κ measurable cardinals. Furthermore, like measurability, supercompactness also has an ultrafilter characterization. Thus supercompact cardinals are thought of as "the proper generalization of measurability" (Solovay, Reinhardt and Kanamori [1978, §2]). The rule of thumb involved here, *generalization*, seems to be a presumption in favor of a natural strengthening of a well-supported axiom. Of course any large cardinal axiom also acquires intrinsic support from *maximize*. (Other rules of thumb favoring the Axiom of Supercompact Cardinals are discussed in VI.2 below.)

Until recently (see VI.4 below), the only significant consequences of the Axiom of Supercompact Cardinals were various relative consistency results. When a statement is too strong to be proved consistent relative to ZFC alone, its consistency can sometimes be derived from the assumption that ZFC plus some further axiom is consistent. (For example, recall Solovay's results from the consistency of "ZFC + The Axiom of Inaccessibles Cardinals" mentioned in [BAI, §III].) Several strong results of this sort follow from the consistency of "ZFC + The Axiom of Supercompact Cardinals" (see e.g. Foreman, Magidor and Shelah [MM]).

Notice that relative consistency results of this sort involving large cardinals are among the most useful applications of these axioms:

> ... large cardinals via the method of forcing turn out to be the natural measures of the consistency strength of ZFC + φ for various statements φ in the language of set theory.

(Kanamori and Magidor [1978, p. 105]) Large cardinals provide such a yardstick because they fit into an ordering:

> As our edifice grew, we saw how one by one the large cardinals fell into place in a *linear* hierarchy. This is especially remarkable in view of the ostensibly

> disparate ideas that motivate their formulation. As remarked by H. Friedman, this hierarchical aspect of the theory of large cardinals is somewhat of a mystery.

(Kanamori and Magidor [1978, p. 264]; see also Parsons [1983, p. 297] and Wang [1974, p. 555]). This unexpected pattern suggests that large cardinal axioms are straightforward ways of saying that the iterative hierarchy contains more and more levels, that is, that they are implementations of *maximize*:

> ... the neat hierarchical structure of the large cardinals and the extensive equi-consistency results that have already been demonstrated to date are strong plausibility arguments for the inevitability of the theory of large cardinals as *the* natural extension of ZFC.

(Kanamori and Magidor [1978, p. 264]) Thus the relative consistency results and the linear ordering of the large cardinal axioms provide extrinsic evidence for the large cardinal program in general.

VI.2. *Arguments for supercompact cardinals.* Two further rule of thumb based arguments have been offered in favor of Supercompact Cardinals. The first is a fairly simple set-theoretic argument based on the model theoretic Vopěnka's principle. The second, via extendibles,[14] is a more elaborate argument due to Reinhardt. As mentioned in §IV above, it depends on somewhat dubious pseudo-*reflection* principles. Vopěnka first.

The most general version of Vopěnka's principle states that any proper class of structures for the same language will contain two members, one of which can be elementarily embedded in the other. The rule of thumb usually cited as lying behind this principle is the idea that the proper class of ordinals is extremely rich (Kanamori and Magidor [1978, p. 196]). Suppose, for example, that a process is repeated once for each ordinal—Ord-many times, we might say—and every step produces a structure. Then *richness* implies that no matter how closely we keep track of the structures generated, there are so many ordinals that some will be indistinguishable.[15] A similar idea can be developed from *reflection*: Anything true of V is already true of some R_α, that is, there is an R_α that resembles V. This property of V should also be reflected, that is, there is an R_α with a smaller R_β that resembles it.

---

[14]For details of extendibles, see Solovay, Reinhardt and Kanamori [1978, §5], or Kanamori and Magidor [1978, §16]. On the relationship between supercompacts and extendibles, Kanamori and Magidor remark:

> All in all, supercompactness and extendibility have similar features ... Supercompactness has the flavor of a generalization from measurability, but extendibility reflects more ethereal ambitions. (pp. 196, 192)

The nature of these "ethereal ambitions" will emerge from Reinhardt's argument, below. As I find this argument flawed, I will keep the emphasis here on supercompacts, rather than extendibles. It is also worth noting that supercompacts seem to occur more naturally in the hypotheses of theorems.

[15]Notice that the thinking behind *richness* is very close to that behind Martin's version of *reflection* in [BAI, §III].

Either way, we get a new rule of thumb, *resemblance*:

> ... there are R_α's that resemble each other.
>
> ... there should be stages R_α and R_β which look very much alike.

(Solovay, Reinhardt and Kanamori [1978, Introduction]; Martin [1976, p. 86]; see also Kanamori and Magidor [1978, p. 104]). The trick, of course, comes in spelling out "resembles".

To do this, let us go back to *richness* and imagine ourselves in an Ord-long process, generating an R_α at each stage, one for each ordinal.[16] Suppose we step several ranks at a time, so that by step α, we are already to R_{γ_α}, for some $\gamma_\alpha > \alpha$. We keep careful track of the structures at each stage by making copious notations on a clipboard, one scoresheet for every stage; we note down every detail of the structure we have just generated, along with every detail of the process that got us there. *Richness* then implies that with so many stages, our scoresheets cannot all be different. At step one, we record the complete diagram of

$$(R_{\gamma_0}, \in, \langle R_{\gamma_\beta}: \beta < 0\rangle).$$

At step two, we look to see if that scoresheet is satisfied by

$$(R_{\gamma_1}, \in, \langle R_{\gamma_\beta}: \beta < 1\rangle).$$

Of course it is not, so we write down the complete diagram of this new structure. And so on. At each step, we generate a new structure, then check to see if any of our old scoresheets will do; if not, we prepare a new one.

Richness then guarantees that we will eventually reach a step α' where one of our old scoresheets will match up. That is, we will reach a step α' where

$$(R_{\gamma_{\alpha'}}, \in, \langle R_{\gamma_\beta}: \beta < \alpha'\rangle)$$

is a model of the complete diagram of

$$(R_{\gamma_\alpha}, \in, \langle R_{\gamma_\beta}: \beta < \alpha\rangle)$$

for some $\alpha < \alpha'$. This means that the smaller structure can be elementarily embedded in the larger; that is:

$$\exists j: (R_{\gamma_\alpha}, \in, \langle R_{\gamma_\beta}: \beta < \alpha\rangle) \xrightarrow{\text{e.e.}} (R_{\gamma_{\alpha'}}, \in, \langle R_{\gamma_\beta}: \beta < \alpha'\rangle).$$

This embedding must be nontrivial, because:

$$\begin{aligned} \alpha' &= \text{length}(\langle R_{\gamma_\beta}: \beta < \alpha'\rangle) \\ &= \text{length}(j(\langle R_{\gamma_\beta}: \beta < \alpha\rangle) \\ &= j(\text{length}(\langle R_{\gamma_\beta}: \beta < \alpha\rangle) \\ &= j(\alpha). \end{aligned}$$

[16] This treatment of the argument from Vopěnka's principle was suggested by Magidor and clarified in discussion with Martin.

We thus have a nontrivial elementary embedding of R_α into $R_{\alpha'}$. Our conclusion is a special case of Vopenka's principle, namely, that in any proper class of R_α's, there is a nontrivial elementary embedding of one into another. We get a supercompact by applying this to the class of all R_α's for limit α that reflect supercompactness. Some large cardinal theorists see this blend of *richness*, *reflection* and *resemblance* as providing strong intrinsic evidence for the larger large cardinal axioms.

Reinhardt's argument also begins with the consideration of the proper class ORD of all ordinals. The universe V is then R_{ORD}. Reinhardt now asks that we look at V and ORD as it were "from the outside", in which case we see that there would be further ordinals and ranks of the form ORD + 1, ORD + ORD, $R_{\text{ORD}+1}$, $R_{\text{ORD}+\text{ORD}}$, and so on. At this point, we seem to have introduced things other than sets, which threatens the universality of set theory, but Reinhardt proposes that we

> ... mitigate this sorrow by seeing the universality of set theory not in the extension of the concept set, but in the applicability of the theory of sets. [1974, p. 198]

In other words, we assume that our theory of sets is *the* universal theory of collections, and hence that it applies to these new objects. This gesture produces lots and lots of these class-like entities, lots of ordinal-like objects greater than ORD, lots of stages of construction after V; and they all obey the axioms of set theory.

This treatment is neat, so neat that we begin to wonder if these new layers really consist of entities of a new and different type; perhaps we just forgot to finish the iterative hierarchy in the first place. To this Reinhardt replies by drawing a distinction between sets and classes that depends on their behavior in counterfactual situations. For example, the set consisting of the current members of congress would be the same in any case, but the class of current members of congress would have been different if the voters had favored the Republicans instead of the Democrats. Reinhardt also imagines that there might be more ordinals in some counterfactual situation, and hence, that there might have been more stages and more sets than there are. Granted this assumption, a set is completely determined by its members—it has the same members in every possible world—but a class might have more members in another possible world—as, for example, the class ORD has more members in a counterfactual situation with more ordinals.

Now let us imagine one of these counterfactual situations, a projected universe with more ordinals. Reinhardt calls these extra ordinals, and the sets in the stages they number, imaginary ordinals and imaginary sets. Our ORD becomes, in the projected world, a new class $j(\text{ORD})$ that consists of real and imaginary ordinals, while the old ORD is just an imaginary set, that is, $j(\text{ORD}) > \text{ORD}$. Sets, on the other hand, do not change their membership in counterfactual situations, so $j(x) = x$, for all $x \in V$.

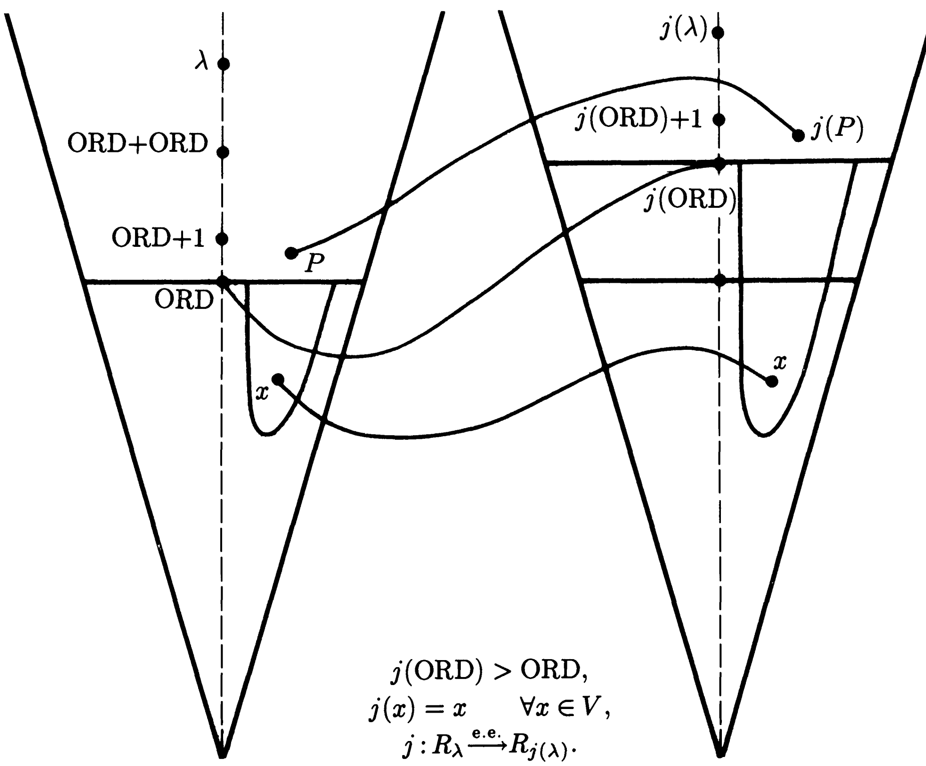
What about truth? Well, consider a proper class P. P consists of sets; it is a subset of R_{ORD}, a member of $R_{\text{ORD}+1}$. Thus $j(P)$, in the projected world, consists of sets and imaginary sets; it is a subset of $R_{j(\text{ORD})}$, and a member of $R_{j(\text{ORD})+1}$. Now set theory is the universal theory of collections, so what is true of P in $R_{\text{ORD}+1}$ should be true of $j(P)$ in $R_{j(\text{ORD})+1}$, that is,

$$j\colon R_{\text{ORD}+1} \xrightarrow{\text{e.e.}} R_{j(\text{ORD})+1}.$$

And the same should be true of the extra layers of proper classes. That is, if λ is some ordinal-like object greater than ORD, then

$$j\colon R_\lambda \xrightarrow{\text{e.e.}} R_{j(\lambda)}.$$

Thus for any ordinal-like λ greater than ORD, we have argued that there is an elementary embedding of R_λ into $R_{j(\lambda)}$ with ORD as critical point, as shown in the figure:



The "real" world The "projected" world

This is just to say that ORD is extendible. All that is needed now is an application of *reflection*: if ORD is extendible, then there should be an extendible cardinal κ. From this we get our supercompact.

This argument has a number of severe shortcomings. The first arises even if we accept Reinhardt's premises; it is internal to the argument itself. Consider once again the purported universality of set theory. This is first applied to guarantee that the theory of the "real" world, with its extra stages, is identical to the theory of V. So far, so good. The second application comes when the "real" world is compared with the projected world. Here it would seem fair to conclude that the theory of the projected world is the same as that of the "real" world, that is, that they are elementarily equivalent. This is enough to assure us an embedding that preserves truth for the definable proper classes, but Reinhardt needs the full force of the elementary embedding j. It is hard to see how the universality of set theory will do this job.

Several other objections arise once we allow ourselves to question Reinhardt's premises. First, there are the alarming entities ORD + 1 and $R_{\mathrm{ORD+ORD}}$. The Vopěnka argument involves thinking of proper classes of sets, but nothing so extravagant and potentially treacherous as these. Second, there is the use of counterfactual situations to distinguish these new entities from sets. I think even those with strong modal intuitions will have trouble imagining how there might be more pure sets and ordinals than there are. After all, V is supposed to contain all the sets and ordinals there could possibly be.

Finally, there is a pernicious ambiguity in Reinhardt's notion of a proper class. Everyone grants that collections can be thought of in two quite different ways: as extensions of concepts on the Fregean model, or as combinatorially generated in stages on the iterative model. These are sometimes called the "logical" and the "mathematical" notion of collection, respectively. When Reinhardt argues that classes differ from sets in their behavior in counterfactual situations, he is playing on the logical notion of extension; the extension of the concept "ordinal" is different in the projected world. On the other hand, when he argues that set theory should apply to all collections, classes included, he is thinking of classes on the mathematical or combinatorial model. On the logical notion of class, there is little reason to think that set theory should apply to entities so different from sets, and abundant reason to think that it should not. For example, it seems that logical classes, like the class of all infinite classes, can be self-membered.[17]

But even if Reinhardt's argument is flawed, we retain the Vopěnka argument based on *richness*, *reflection* and *resemblance*, as well as the earlier defenses in terms of *maximize* and *generalize*.

VI.3. *Huge cardinals and beyond.* If strengthening the closure condition on the range of the elementary embedding gives a natural *generalization* of measurability, *generalization* itself suggests the closure conditions might be strengthened even further. Indeed, why should the range of the elementary embedding not be completely closed, that is, why should it not be V itself?:

> In the first flush of experience with these ideas, Reinhardt speculated on the possibility of an ultimate extension: *Could there be an elementary embedding* $j\colon V \to V$?

(Kanamori and Magidor [1978, p. 200]) It was not long before Kunen destroyed this hope. In [1971], he showed that if j is an elementary embedding of V into a transitive M, if κ is the critical point of j, and if $j(\kappa) = \kappa_1$, $j(\kappa_1) = \kappa_2, \ldots$ and $\kappa_\omega = \lim_{n\to\infty} \kappa_n$, then there is a subset of κ_ω that is not in M. Thus M is not V.

[17] Some aspects of the set/class distinction are discussed in my [1983].

Kunen's theorem shows when a large large cardinal axiom is too large, so large that it contradicts ZFC (specifically, the Axiom of Choice):

> Kunen's result will limit our efforts in that we cannot embed the universe into too "fat" an inner model.

(Solovay, Reinhardt and Kanamori [1978, §1]) Modern set theorists have reacted much as Zermelo did to the inconsistencies of his day, that is, by applying the rule of thumb *one step back from disaster* (see [BAI, §1.4]). Thus they consider n-huge cardinals:

> ... they assert stronger and stronger closure properties, until their natural ω-ary extension is inconsistent.

(Kanamori and Magidor [1978, p. 202]) A cardinal κ is *n-huge* iff there is an elementary embedding j of V into a transitive M such that M is closed under arbitrary sequences of length κ_n. Kunen's theorem says that there is no such thing as an ω-huge cardinal.

Notice that 0-hugeness is just measurability. In addition:

> The n-huge cardinals certainly have an analogous flavor to λ-supercompact cardinals, but there is an important difference: While λ-supercompactness is hypothesized with an *a priori* λ in mind as a proposed degree of closure for M, n-hugeness has closure properties only *a posteriori*: M here is to be closed under κ_n-sequences, however large the κ_n turn out to be. This is a strengthening of an essential kind.

(Kanamori and Magidor [1978, p. 198]) Thus the comparison with supercompacts tends to tarnish the image of the n-huge cardinals:

> Indeed, it is not clear how to motivate n-hugeness ... at all.

(Kanamori and Magidor [1978, p. 198]) But if intrinsic support is lacking, at least n-huge cardinals do have a familiar sort of ultrafilter characterization, and they have played a role in some relative consistency results. Both these are cited as weak extrinsic evidence (Kanamori and Magidor [1978, pp. 198, 200]).

Another method of applying *one step back from disaster* is suggested by the form of Kunen's proof. The argument depends on the occurrence of a certain function in the domain of the elementary embedding. The domain of the function is the set of ω-sequences from κ_ω, so the function itself first occurs at the level $R_{\kappa_\omega+2}$. Thus Kunen actually shows that there is no nontrivial elementary j: $R_{\kappa_\omega+2} \to R_{\kappa_\omega+2}$. This leaves two possible forms of "there is a nontrivial elementary embedding of some R_λ into itself":

EE(I) $\qquad \exists j$: $R_{\kappa_\omega+1} \xrightarrow{\text{e.e.}} R_{\kappa_\omega+1}$;

EE(II) $\qquad \exists j$: $R_{\kappa_\omega} \xrightarrow{\text{e.e.}} R_{\kappa_\omega}$.

It is known that EE(I) implies EE(II) and that EE(II) implies the existence of n-huge cardinals for every n. Indeed, the large cardinal property which EE(I) asserts of the critical point of its embedding is so strong that the existence of such a cardinal implies the existence of an even larger cardinal with the same property (Kanamori and Magidor [1978, p. 203]).

Even the defenders of large large cardinals express discomfort over axioms this strong:

> It seems likely that [EE(I) and EE(II) are] inconsistent since they appear to differ from the proposition proved inconsistent by Kunen in an inessential technical way. The axioms asserting the existence of [n-huge] cardinals, for $n > 1$, seem (to our unpracticed eyes) essentially equivalent in plausibility: far more plausible than [EE(II)], but far less plausible than say extendibility.

(Solovay, Reinhardt and Kanamori [1978, §7]; see also Kanamori and Magidor [1978, p. 202]). Notice also that, given the reformulation of Kunen's result, EE(I), if consistent, would seem to be the largest possible large cardinal axiom. Some set theorists feel that for every large cardinal axiom there should be a larger, and this sentiment counts for them against EE(I).

VI.4. *Connections with Determinacy.* Recall that the existence of a measurable cardinal implies the determinacy of Σ^1_1 sets of reals (see V.2). This sort of result was extended one level further, to the determinacy of Σ^1_2 sets, by Martin in [1978], but the large cardinal axiom this proof requires is EE(I). This result caused some soul-searching among those who had hoped to increase the intrinsic support of determinacy hypotheses by deriving them from large cardinal axioms, but who also felt uncomfortable with EE(I). Furthermore, if the "last" large cardinal axiom was indeed necessary to prove $\mathrm{Det}(\Sigma^1_2)$, then the program of proving all of QPD from such axioms seemed hopeless. Still, the fact that EE(I) implied more determinacy, and the naturalness of the proof, led to something of a softening in the attitude towards this axiom.

There the situation remained until 1984, when consideration of the sets constructible from $R_{\kappa_\omega+1}$ led Woodin to an elementary embedding condition between EE(I) and Kunen's inconsistency:

EE(0) $$\exists j\colon L[R_{\kappa_\omega+1}] \xrightarrow{\text{e.e.}} L[R_{\kappa_\omega+1}].$$

Then came the result that everyone had been hoping for; Woodin went on to derive the full QPD from EE(0). With the discovery of EE(0), EE(I) no longer seemed the "last" large cardinal axiom, and EE(0) produced a natural and detailed theory of $L[R_{\kappa_\omega+1}]$ that resembled the theory of $L[R]$ on the assumption QPD. All this was counted as extrinsic evidence in their favor.

Recall that in the wake of Martin's earlier theorem deriving $\mathrm{Det}(\Sigma^1_1)$ from the existence of a measurable cardinal, various determinacy assumptions were proved equivalent to the inner model versions of the corresponding large cardinal axiom (see V.2). In addition to indiscernibles and a slightly smaller large cardinal, these inner models have well-orderings of the reals that are as simple as their level of determinacy allows. For example, the existence of the sharps (the inner model version of the Axiom of One Measurable Cardinal) guarantees the existence of an inner model of ZFC with indiscernibles; that model has Δ^1_1 determinacy, so it cannot have a Δ^1_1 well-ordering of the reals, but it does have a Δ^1_2 well-ordering. Similarly, the inner model of ZFC + 2MC (the inner model version of the Axiom of Three Measurable Cardinals) has $(\omega^2 + 1) - \Pi^1_1$ determinacy, so it cannot have a Δ^1_2 well-ordering of a certain special sort, but it does have a Δ^1_3 well-ordering. If this pattern were to continue, as most set theorists concerned with the problem expected that it would, then there should be inner models of all large cardinal axioms up to EE(I) with various degrees of Δ^1_2 determinacy and Δ^1_3 well-orderings of the reals. Alas, the inner model theorists, Mitchell, Dodd, Steel and others, were unable to find such a model; their efforts failed before they reached a supercompact cardinal.

The reasons for this failure were soon clarified from another quarter. Working on the development of further relative consistency results, Foreman, Magidor and Shelah were able to improve an older result of Kunen's by reducing the hypothesis from the consistency of a huge cardinal to the consistency of a supercompact cardinal (see [MM]). Shelah and Woodin then managed to reduce the hypothesis even more, to something between measurable and supercompact, and along the way, Woodin realized their methods led to another surprising result: if there is a supercompact cardinal, then every quasiprojective set of reals is Lebesgue measurable, has the Baire and perfect subset properties, and so on. Thus, the model the inner model theorists were seeking—an inner model with a supercompact cardinal and a Δ^1_3 well-ordering of the reals—does not exist. Indeed there is no inner model with a supercompact cardinal and *any* quasiprojective well-ordering of the reals. The neat inner model theory that did so much to familiarize measurable cardinals cannot be duplicated for supercompacts.[18]

But what about determinacy? There were two possibilities. Up to this point, the old-fashioned regularity properties like Lebesgue measurability had done hand-in-hand with determinacy. Now the determinacy of quasiprojective sets seemed to require the somewhat staggering assumption of EE(0), while their other regularity properties required only a supercompact cardinal. The first possibility was that determinacy and Lebesgue measurability do in fact diverge here, and the inner model equivalences possible within Δ^1_2 cannot be extended. The second possibility was that QPD could actually be proved from the far weaker assumption of a supercompact cardinal.

Two who believed in the second possibility were Martin and Steel. Woodin had shown that his theorem on the Lebesgue measurability of quasiprojective sets could actually be derived from a complex hypothesis slightly weaker than the existence of a full supercompact cardinal, so Martin and Steel felt they had an exact formulation of the hypothesis that should yield QPD. Further, Steel had extensive experience with the sort of phenomena that had blocked the development of the inner model theory before it was known to be impossible. He and Martin theorized that whatever blocked the construction of a nice inner model might be closely connected with determinacy. Reasoning in this way, they were able to prove PD, and (using another result of Woodin) QPD, from Woodin's hypothesis, and hence, from the existence of a supercompact cardinal.

[18] As the existence of simple inner models made some set theorists more comfortable with measurable cardinals, the nonexistence of such inner models makes supercompacts appear more mysterious, perhaps even dangerous.

This sudden and unexpected reduction in the ante required for QPD naturally contributes strongly to the attractiveness of the theory. All the determinacy needed for descriptive set theory can be viewed as a theorem of "ZFC + The Axiom of Supercompact Cardinals". Indeed, the theory of $L[R]$ under these axioms is in some sense "complete": it is invariant under most forcing extensions (see [MM]). Thus supercompact cardinals gain a tremendous amount of extrinsic evidence, and QPD inherits various intrinsic and rule of thumb support (*maximize*, *generalize*, *richness* and *reflection*) from the Axiom of Supercompact Cardinals. And both are extrinsically supported by the impressive strength of their intertheoretic connections. Thus it is not surprising that some Cabal members view the Martin/Steel theorem as proving the detailed descriptive set theory described in Moschovakis's book [1980].

Of course, now that QPD is seen to follow from the existence of a supercompact cardinal, the Levy/Solovay theorem of [1967] immediately implies that QPD cannot decide the size of the continuum. The next step would be to investigate hypotheses on the structure of $L[\mathscr{P}(R)]$.

§VII. Concluding philosophical remarks. As this is not a history paper, and even more obviously not a logic paper, I feel I owe at least a few philosophical reflections. Of course the motivating force behind the presentation of all this material has been a philosophical one: I hope to display the role of nondemonstrative arguments in set theory, especially in the search for new axioms, and to pose the philosophical task, for epistemologists and philosophers of mathematics, of describing and accounting for this role. In this final section, I will summarize and lightly categorize the data, then address a few random remarks to the serious philosophical questions raised.

The defenses given here for set-theoretic axiom candidates have been roughly divided into three categories: intrinsic, extrinsic and rules of thumb. So far, I have not tried to classify particular rules of thumb as intrinsic, extrinsic or other, but it should be clear that there is considerably variation within that group. Let me begin with a rather stylized discussion of intrinsic justification.

I have argued elsewhere [1980] that we acquire our most primitive physical and set-theoretic beliefs when we learn to perceive individual objects and sets of these. We come to believe, for example, that objects do not disappear when we are not looking at them, and that the number of objects in a set does not change when we move the objects around. These intuitive beliefs are not incorrigible—consider, for example, our erstwhile convictions that objects are solid, or that every property determines a set—but they do provide a starting point for our physical and mathematical sciences. The simplest axioms of set theory, like Pairing, have their source in this sort of intuition. If they are not strictly part of the concept (whatever that comes to), they are acquired along with the concept. Given its origin in prelinguistic experience, the best indication of intuitiveness is when a claim strikes us as obvious, or in Gödel's words, when the axioms "force themselves upon us as being true" [1947/64, p. 484].

The extrinsic evidence cited in previous sections came in a bewildering variety of forms, among them: (1) confirmation by instances (the implication of known lower-level results, as, for example, *reflection* implies weaker reflection principles known to be provable in ZFC); (2) prediction (the implication of previously unknown lower level results, as, for example, the Axiom of Measurable Cardinals implies the determinacy of Borel sets which is later proved from ZFC alone); (3) providing new proofs of old theorems (as, for example, game-theoretic methods give new proofs of Solovay's older set-theoretic results); (4) unifying new results with old, so that the old results become special cases of the new (as, for example, the proof of PWO(Π^1_1) becomes a special case of the periodicity theorem); (5) extending patterns begun in weaker theories (as, for example, the Axiom of Measurable Cardinals allows Souslin's theorem on the perfect subset property to be extended from Σ^1_1 to Σ^1_2); (6) providing powerful new ways of solving old problems (as, for example, QPD settles questions left open by Lusin and Souslin); (7) providing proofs of statements previously conjectured (as, for example, QPD implies there are no definable well-orderings of the reals); (8) filling a gap in a previously conjectured "false, but natural proof" (as, for example, Det(Δ^1_2) filled the gap in Moschovakis's erroneous "sup" proof of PWO(Π^1_3)); (9) explanatory power (as, for example, Silver's account of the indiscernibles in L provides an explanation of how and why $V \neq L$); (10) intertheoretic connections (as, for example, the connections between determinacy hypotheses and large cardinal assumptions count as evidence for each).

All of these more or less correspond to forms of confirmation recognized in the physical sciences. I would like very much to give an account of their rationality, but even our best philosophers of science, from Hempel [1945] to Glymour [1980], have so far been satisfied with predominantly descriptive accounts. A careful analysis of the structure of such arguments must precede what we hope will be an explanation of why they lead us toward truth (cf. Glymour [1980, p. 238]).

Finally, rules of thumb. When uncritical, intuitive work with sets was interrupted by the appearance of the paradoxes, examination of previously unexamined practice revealed that full Comprehension was not in fact used. Rather, sets were thought of as being formed from objects already available. This lead to the separation of sets from classes, and eventually, to the development of the rule *iterative conception.* The source of this rule of thumb in pretheoretical practice, and the overwhelming impression of its naturalness once it was specified, suggest that its origin is at least partly intuitive (see, e.g. Shoenfield [1967, p. 238]). *Realism*, *maximize*, and its companion, *richness*, are all closely tied to *iterative conception.* Finally, *reflection* is often claimed to be intuitive, perhaps with grounds in *maximize* as well. *Inexhaustibility* is just a special case of *reflection*, and *resemblance* is a consequence.

In contrast, the evidence for the boldest of our rules of thumb—*Cantorian finitism*—is predominantly extrinsic, lying in the depth, breadth and effectiveness of the subject it launched. Other rules have the flavor of general methodological maxims, principles that express our higher-order preferences for theories of one sort or another. An example from physical science is Maxwell's principle, which states that a law of nature should be valid at all points in space and time (see Wilson [1979] for discussion). *Diversity* and *generalization* are rules of thumb at a similar level of abstraction. *One step back from disaster*, and its special case *limitation of size*, might also be viewed as methodological, though they share something of the spirit of *maximize. Banishment*, on the other hand, seems neither intrinsic, nor extrinsic, nor methodological, but rather based in seat-of-the-pants experience with the theory in question, like most conjectures.

Finally, *uniformity* and its capriciousness companion *whimsical identity* have been defended both as methodological principles akin to Maxwell's—a good theory does not single out particular locations—and as intuitions about the nature of the iterative hierarchy connected with *richness* and *resemblance*. Either way, we have seen the dangers inherent in applications of these two related rules. Perhaps what is needed is a theory of exactly what sorts of properties are allowable in *uniformity* and *whimsical identity* arguments, much as only so-called "structural" properties are allowed in *reflection* arguments.[19] Another possibility would be to grant evidential status to *uniformity* and *whimsical identity* arguments only in the presence of good evidence for consistency, or perhaps to relegate them to the status of heuristic devices for generating hypotheses that must then be justified by other, probably extrinsic, means.[20]

If, as we have seen, the practice of mathematics can be understood as analogous to that of the physical sciences in a great many respects, it must also be admitted that there is a striking difference: mathematicians rarely rely on observations in their nondemonstrative testing. This can be understood if we revert to our perceptual story. When we learn to see sets of things, we learn to see number properties, and from this we develop the humblest of our mathematical sciences: arithmetic. If our rudimentary physical science is the study of things qua stuff, arithmetic is the study of things qua individuals, the study of sets of things, and as such it is independent of the make-up of a set's elements qua stuff. As far as arithmetic is concerned, the particular things in its sets are irrelevant, as is their stuff; a set of a given cardinality is interchangeable, for arithmetical purposes, with a wide range of others, sets with different particular elements but the same cardinality, even sets of symbols.[21] Thus, once our perceptual relation to the physical world has produced our ability to see sets and our basic intuitions about them, the further observation of particulars is unimportant.

This is not to say that the physical world remains entirely irrelevant after this initial stage, but before I mention its further incursions, I should say a word about the ramifications of a naturalistic, empirical, perception-based account of mathematical knowledge.[22] Such views face an unavoidable challenge from the venerable philosophical observation that while our various perceptual, neurological and

[19] We might try something along the following lines: in both forms of argument, the crucial property, the one that is to recur or the one that appears in the whimsical identity, must be "natural". Obviously a natural property cannot involve notions like "first" or "smallest", and it cannot involve proper names. After this, it is hard to know what to say, except that the failures of *uniformity* and *whimsical identity* could be explained away as involving instances of unnatural properties. That is, for example, "2 is *the* even prime" shows that "prime" would be more naturally defined as "odd number not divisible by anything but itself and 1", and "$\aleph_0$ is *the* cardinal κ such that $\forall n, m < \kappa\ \kappa \to \kappa^n_m$" shows that arrow properties should not be formulated to allow infinite exponents.

[20] Representatives of each these various opinions on *uniformity* and *whimsical identity* arguments can be found within the Cabal.

[21] Even sets of appearances. This is why the threat of sensory illusion is less pressing for mathematics than for physical science: even if there were an evil demon systematically deceiving us as to the structure of the external world, arithmetic would still apply to the world of appearances.

[22] The idea that perception is involved in the genesis of mathematical knowledge is fairly popular these days. See, for example, Resnik [1982], Kitcher [1983], Parsons [1979–80].

evolutionary interactions with the world might well tell us what is true, they cannot tell us what must be true. This, coupled with the equally venerable assumption that mathematical truths are necessary, creates a mystery. We begin to ask ourselves odd questions: if our world (or the evil demon's illusion) were different, would we have a different arithmetic? Of course, it is much easier to imagine a world with a different physical make-up than ours, or even different physical laws, than to imagine one to which our arithmetic does not apply. But then again, if objects systematically appeared and disappeared during counting, perhaps we would calculate differently; at least it seems likely that the ancient Babylonians (or whoever) would have lost interest in the subject. Still, it seems that once a world has two objects, it has a potential infinity of which arithmetic is true: the apple, the orange, the set of these two, the set of the preceding three, etc.[23] Perhaps only a world with absolutely no differentiation, a world completely homogeneous, the eternal oneness of the mystics, would be without number properties. But even if we leave aside the irritating inconclusiveness of musings along these lines, I think we must question their moral, their importance, their significance. We lack so much as a clear understanding of what it means to say that something is necessary: true in all possible worlds? true due to some irreducibly modal property of this world? At this point, it seems to me that the most reasonable answer to the old question—how do we know that mathematical truths are necessary?—must be that we do not know.[24]

It is worth noting that the same goes for certainty. This obvious point should not need belaboring, except when a mathematical epistemologist attempts to find arguments strong enough to "convince the skeptic". Philosophers gave up the search for such arguments in natural science long ago; its retention in the philosophy of mathematics can only be traced to an outmoded vision of the nature of mathematical knowledge. No one would expect even the best scientific arguments to be absolutely justifying. Our epistemological inquiries in mathematics will be hampered if we set an unreasonably high standard.

What, then, is the post-perceptual evidential connection of set theory with the physical world? I would suggest that it is the profound applicability of set theory's twin pillars: number theory and geometry/analysis. While number theory has its origin in counting, geometry arises from the study of the shapes of things (things as individuated objects, that is, not as amorphous arrays of physical stuff) and analysis from the study of their motions. Set theory systematizes and explains these two extravagantly useful branches of mathematics, and in so doing, gains much of its own justification (recall the extrinsic argument for the Power Set Axiom in [BAI, §I.6]). Notice that the continuum problem, whose independence prompted the search for new axioms, and whose solution would provide the most impressive extrinsic evidence, is itself a question about the real numbers of physical science. This is a central reason why many set theorists are confident of its meaningfulness, and thus of the propriety of the search for new axioms herein described.

[23] Perhaps this is the truth behind Brouwer's obscure "two-oneness". See his [1912]. In any case, one object would do, as long as it was differentiated from its background: the it and the not-it.

[24] I do not mean by this that we know mathematics to be contingent, either, but that we have no dependable information whatever on the question (assuming it is well-formed).

The success of set theory—its objectivity and its applicability—confirm the enterprise and its justificatory practices as a whole, but within that whole, the particular methods can be analyzed, supported or criticized individually. Not only would a clear account of the structure and rationality of nondemonstrative set theoretic arguments provide solace for the practitioners and philosophers of the subject, but it might even help with the very real problem of locating new rules of thumb and new axiom candidates for the solution of the continuum problem. I should emphasize that this is not a project of importance only to those with a Platonistic bent. It is central to any philosophical position for which the size of the continuum is a real issue: all realistic philosophies of set theory, even those that eschew mathematical objects (like Kitcher's [1983], Resnik's [1981], [1982], or Shapiro's [1983]); modalist accounts that depend on full second-order models (like Putnam's [1967] and Hellman's [1986]); and even some versions of Field's nominalism (the second-order option of [1985] where only one of "ZFC + $(V = L)$" and "ZFC + QPD" can be conservative (see his footnote 16)). This strongly suggests that in this area at least, we would do well to drop the ingrained philosophical tendency to concentrate of the differences (however minute) between positions, and to engage in a cooperative effort.

REFERENCES

J. W. ADDISON

[1959] *Some consequences of the axiom of constructibility*, ***Fundamenta Mathematicae***, vol. 46 (1959), pp. 337–357.

[1974] *Current problems in descriptive set theory*, in Jech [1974], pp. 1–10.

J. W. ADDISON and Y. N. MOSCHOVAKIS

[1968] *Some consequences of the axiom of definable determinateness*, ***Proceedings of the National Academy of Sciences of the United States of America***, vol. 59 (1968), pp. 708–712.

Y. BAR-HILLEL, editor

[1970] ***Mathematical logic and foundations of set theory***, North Holland, Amsterdam, 1970.

J. BARWISE, editor

[1977] ***Handbook of mathematical logic***, North-Holland, Amsterdam, 1977.

P. BENACERRAF and H. PUTNAM, editors

[1983] ***Philosophy of mathematics***, 2nd ed., Cambridge University Press, Cambridge. 1983.

D. BLACKwell

[1967] *Infinite games and analytic sets*, ***Proceedings of the National Academy of Sciences of the United States of America***, vol. 58 (1967), pp. 1836–1837.

L. E. J. BROUWER

[1912] *Intuitionism and formalism*, in Benacerraf and Putnam [1983], pp. 77–89

M. DAVIS

[1964] *Infinite games of perfect information*, ***Advances in game theory*** (M. Dresher et al., editors), Annals of Mathematics Studies, vol. 52, Princeton University Press, Princeton, New Jersey, 1964, pp. 85–101.

J. E. FENSTAD

[1971] *The axiom of determinateness*, ***Proceedings of the second Scandinavian logic symposium*** (J. F. Fenstad, editor), North-Holland, Amsterdam, 1971, pp. 41–61.

H. FIELD

[1985] *On conservativeness and incompleteness*, ***Journal of Philosophy***, vol. 82 (1985), pp. 239–260.

M. FOREMAN, M. MAGIDOR, and S. SHELAH

[MM] *Martin's Maximum, saturated ideals, and nonregular ultrafilters* (to appear).

D. GALE AND F. M. STEWART

[1953] *Infinite games with perfect information*, **Contributions to the theory of games**, vol. 2, Annals of Mathematics Studies, vol. 28, Princeton University Press, Princeton, New Jersey, 1953, pp. 245–266.

C. GLYMOUR

[1980] **Theory and evidence**, Princeton University Press, Princeton, New Jersey, 1980.

K. GÖDEL

[1947/64] *What is Cantor's continuum problem?*, in Benacerraf and Putnam [1983], pp. 470–485.

L. HARRINGTON

[1978] *Analytic determinacy and* $0^{\#}$, this JOURNAL, vol. 43 (1978), pp. 685–693.

G. HELLMAN

[1986] *A modal interpretation of mathematics*, circulated photocopy.

C. HEMPEL

[1945] *Studies in the logic of confirmation*, **Mind**, vol. 54 (1945), pp. 1–26, 97–121; reprinted in C. W. Hempel, **Aspects of scientific explanation and other essays in the philosophy of science**, MacMillan, 1965, pp. 3–46.

T. J. JECH, editor

[1974] **Axiomatic set theory**, Proceedings of Symposia in Pure Mathematics, vol. 13, part II, American Mathematical Society, Providence, Rhode Island, 1974.

K. KANAMORI and M. MAGIDOR

[1978] *The evolution of large cardinal axioms in set theory*, Higher set theory (G. H. Müller and D. S. Scott, editors), Lecture Notes in Mathematics, vol. 669, Springer-Verlag, Berlin, 1978, pp. 99–275.

P. KITCHER

[1983] **The nature of mathematical knowledge**, Oxford University Press, Oxford, 1983.

K. KUNEN

[1971] *Elementary embeddings and infinitary combinatorics*, this JOURNAL, vol. 36 (1971), pp. 407–413.

A. LEVY and R. M. SOLOVAY

[1967] *Measurable cardinals and the continuum hypothesis*, **Israel Journal of Mathematics**, vol. 5 (1967), pp. 234–248.

P. MADDY

[1980] *Perception and mathematical intuition*, **Philosophical Review** vol. 89 (1980), pp. 163–196.

[1983] *Proper classes*, this JOURNAL, vol. 48 (1983), pp. 113–139.

[1984] *New directions in the philosophy of mathematics*, **PSA 1984** (P. Kitcher, editor), Philosophy of Science Association, 1985, pp. 425–447.

[BAI] *Believing the axioms*. I, this JOURNAL, vol. 53 (1988), pp. 481–511.

D. A. MARTIN

[PSCN] *Projective sets and cardinal numbers*, circulated photocopy.

[1968] *The axiom of determinateness and reduction principles in the analytic hierarchy*, **Bulletin of the American Mathematical Society**, vol. 74 (1968), pp. 687–689.

[1970] *Measurable cardinals and analytic games*, **Fundamenta Mathematicae**, vol. 66 (1970), pp. 287–291.

[1975] *Borel determinacy*, **Annals of Mathematics**, ser. 2, vol. 102 (1975), pp. 363–371.

[1976] *Hilbert's first problem: the continuum hypothesis*, **Mathematical developments arising from Hilbert problems** (F. E. Browder, editor), Proceedings of Symposia in Pure Mathematics, vol. 28, American Mathematical Society, Providence, Rhode Island, 1976, pp. 81–92.

[1977] *Descriptive set theory*, in Barwise [1977], pp. 783–815.

[1978] *Infinite games*, **Proceedings of the International Congress of Mathematicians (Helsinki, 1978)**, vol. 1, Academia Scientiarum Fennica, Helsinki, 1980, pp. 269–273.

Y. N. MOSCHOVAKIS

[1970] *Determinacy and prewellorderings of the continuum*, in Bar-Hillel [1970], pp. 24–62.

[1980] **Descriptive set theory**, North-Holland, Amsterdam, 1980.

J. MYCIELSKI

[1964] *On the axiom of determinateness*, **Fundamenta Mathematicae**, vol. 53 (1964), pp. 205–224.

[1966] *On the axiom of determinateness*. II, ***Fundamenta Mathematicae***, vol. 59 (1966), pp. 203–212.

J. MYCIELSKI and H. STEINHAUS

[1962] *A mathematical axiom contradicting the axiom of choice*, ***Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques***, vol. 10 (1962), pp. 1–3.

J. MYCIELSKI and S. SWIERCZKOWSKI

[1964] *On Lebesgue measurability and the axiom of determinateness*, ***Fundamenta Mathematicae***, vol. 54 (1964), pp. 67–71.

J. B. PARIS

[1972] $ZF \vdash \Sigma^0_4$ *determinateness*, this JOURNAL, vol. 37 (1972), pp. 661–667.

C. PARSONS

[1978/80] *Mathematical intuition*, ***Proceedings of the Aristotelian Society***, vol. 80 (1979–1980), pp. 142–168.

[1983] ***Mathematics in philosophy***, Cornell University Press, Ithaca, New York, 1983.

H. PUTNAM

[1967] *Mathematics without foundation*, in Benacerraf and Putnam [1983], pp. 295–311.

W. N. REINHARDT

[1974] *Remarks on reflection principles, large cardinals and elementary embeddings*, in Jech [1974], pp. 189–205.

Set existence principles of Shoenfield, Ackermann, and Powell, ***Fundamenta Mathematicae***, vol. 84 (1974), pp. 5–34.

M. RESNIK

[1981] *Mathematics and a science of patterns: ontology and reference*, ***Noûs***, vol. 15 (1981), pp. 529–550.

[1982] *Mathematics as a science of patterns: epistemology*, ***Noûs***, vol. 16 (1982), pp. 95–105.

S. SHAPIRO

[1983] *Mathematics and reality*, ***Philosophy of Science***, vol. 50 (1983), pp. 523–548.

J. R. SHOENFIELD

[1967] ***Mathematical logic***, Addison-Wesley, Reading, Massachusetts, 1967.

W. SIERPINSKI

[1925] *Sur une classe d'ensembles*, ***Fundamenta Mathematicae***, vol. 7 (1925), pp. 237–243.

R. M. SOLOVAY

[1969] *The cardinality of* Σ^1_2 *sets*, ***Foundations of Mathematics*** (J. J. Bulloff et al., editors), Symposium papers commemorating the sixtieth birthday of Kurt Gödel, Springer-Verlag, Berlin, 1969, pp. 58–73.

R. M. SOLOVAY, W. N. REINHARDT, and A. KANAMORI

[1978] *Strong axioms of infinity and elementary embeddings*, ***Annals of Mathematical Logic***, vol. 13 (1978), pp. 73–116.

H. WANG

[1974] *The concept of set*, in Benacerraf and Putnam [1983], pp. 530–570.

M. WILSON

[1979] *Maxwell's condition—Goodman's problem*, ***British Journal for the Philosophy of Science***, vol. 30 (1979), pp. 107–123.

R. WOLF

[1985] *Determinateness of certain almost Borel games*, this JOURNAL, vol. 50 (1985), pp. 569–579.

P. WOLFE

[1955] *On the strict determinacy of certain infinite games*, ***Pacific Journal of Mathematics***, vol. 5 (1955), pp. 841–847.

DEPARTMENT OF PHILOSOPHY
UNIVERSITY OF CALIFORNIA AT IRVINE
IRVINE, CALIFORNIA 92717