COMPLEX ANALYSIS
WORKSHEET 3
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1. Set $$f(z)=\frac{e^{iz}}{1+z^2},\ z\in\mathbb{C}.$$

(i) Prove that
$$\lim_{R\to+\infty}\int_{\gamma_R} f(z)dz=0,$$
where $\gamma_R(t)=Re^{it},\ t\in[0,\pi],\ R>0.$

(ii) Compute
$$\int_{-\infty}^{+\infty}\frac{\cos x}{1+x^2}dx.$$

[Hint: Cauchy integral formula for f on the closed curve $\underline{\gamma_R+[-R,R]}$, $R>0$.]

2. Compute
$$\int_\gamma \frac{e^z}{z(1-z)^2}dz,$$
where:

(i) $\gamma(t)=\frac{1}{2}e^{it},\ t\in[0,2\pi]$ (ii) $\gamma(t)=1+\frac{1}{2}e^{it},\ t\in[0,2\pi]$.

3. Show that
$$\int_0^{2\pi}\frac{dt}{4\cos^2 t+9\sin^2 t}=\frac{\pi}{3},$$
by integrating $1/z$ on the curve $\gamma(t)=2\cos t+3i\sin t,\ t\in[0,2\pi]$.

4. Integrating $f(z)=e^z/z$ on the positively oriented circle $|z|=1$, prove: $$\int_0^{2\pi}e^{\cos t}\cos(\sin t)dt=2\pi,\qquad \int_0^{2\pi}e^{\cos t}\sin(\sin t)dt=0.$$

5. Let $z_0\in\mathbb{C}$, $\operatorname{Im}(z_0)<0$, $R>0$ & γ_R the half circle $\gamma_R(t)=Re^{it},\ t\in[\pi,\ 2\pi]$. Prove:

(i)
$$\lim_{R\to+\infty}\int_{\gamma_R}\frac{dz}{z(z-z_0)}=0,\qquad \lim_{R\to+\infty}\int_{\gamma_R}\frac{dz}{z-z_0}=\pi i.$$

(ii)
$$\lim_{R\to+\infty}\int_{-R}^{R}\frac{dt}{t-z_0}.$$

6. Let f be holomorphic on an open set $U\supset D=\{z\in\mathbb{C}:|z|\le 1\}$, $|z_0|<1$ and $\gamma(t)=e^{it},\ t\in[0,2\pi]$. Prove:
$$f(z_0)=\frac{1}{2\pi i}\frac{1}{1-|z_0|^2}\int_\gamma f(z)\frac{1-z\overline{z_0}}{z-z_0}dz,\qquad |f(z_0)|\le\frac{1}{2\pi}\frac{1}{1-|z_0|^2}\int_0^{2\pi}\left|f(e^{it})\right|dt.$$

7. Let f be holomorphic on an open set $U \supset D = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\gamma(t) = e^{it},\ t \in [0, 2\pi]$.

(i) Prove:

$$\int_\gamma \overline{f(z)}dz = 2\pi i \overline{f'(0)}.$$

(ii) Compute:

$$\int_\gamma \overline{z \cos z}dz.$$

8. Let f be holomorphic on an open set $U \supset D[0, R] = \{z \in \mathbb{C} : |z| \leq R\},\ R > 0$. If $f(z_0) = 0$, for some z_0 s.t. $|z_0| < R$, prove:

$$|f(0)| \leq \frac{M_R |z_0|}{R - |z_0|},$$

where $M_R = \max\{|f(z)| : |z| = R\}$.

9. Let f be an entire function s.t.

$$|f(z)| \leq a|z|^2 + b, \quad \forall\, z \in \mathbb{C},$$

where $a,\ b \in (0, +\infty)$. Prove:

(i) for all $n \in \mathbb{N},\ R > 0$,

$$\left|f^{(n)}(0)\right| \leq n! \frac{aR^2 + b}{R^n}.$$

(ii) $\exists\ A, B, C \in \mathbb{C}$ s.t. $|A| \leq a,\ |C| \leq b$ and

$$f(z) = Az^2 + Bz + C, \quad \forall\, z \in \mathbb{C}.$$

10. Let $P(z)$ be a polynomial of degree $n \geq 2$ with $P(z) = a_n z^n + \cdots$ $(a_n \neq 0)$.

(i) Show that

$$\lim_{|z|\to\infty} \left|\frac{P(z)}{a_n z^n} - 1\right| = 0.$$

Conclude that $\exists\ R_0 > 0$ s.t. $\forall : |z| > R_0$,

$$|P(z)| > \frac{|a_n|}{2}|z|^n.$$

(ii) Prove:

$$\lim_{R\to+\infty} \int_{\gamma_R} \frac{1}{P(z)}dz = 0,$$

where $\gamma_R(t) = Re^{it},\ t \in [0, 2\pi],\ R > 0$.

(iii) If γ is a closed simple piecewise smooth curve containing all the roots of $P(z)$, show that

$$\int_\gamma \frac{1}{P(z)}dz = 0.$$

11. Find Taylor's expansion of f around z_0, in each of the following cases:

(i) $f(z) = 1 - \dfrac{2}{1+z} + \dfrac{1}{(1+z)^2}$, $z_0 = i$.

(ii) $f(z) = (\cos z)^2$, $z_0 = \pi$.

12. Find Taylor's expansion of $f(z) = \dfrac{z}{2}(e^{z^2} - e^{-z^2})$ around $z_0 = 0$, and also the derivative $f^{(23)}(0)$.

13. Find Taylor's expansion of $f(z) = \dfrac{z^5}{1+z^4}$ around $z_0 = 0$, and also the derivative $f^{(21)}(0)$.

14. For $|z| < 1$, compute: $\sum_{n=1}^{\infty} n^2 z^n$.

15. Compute the limits

$$\lim_{z\to 0} \frac{(1-\cos z)^2}{(e^z - 1 - z)\sin^2 z}, \qquad \lim_{z\to 0} \frac{1-\cos(2z)}{(e^{2iz}-1)\sin z}.$$

[Hint: Use Taylor's expansions around 0.]

16. Let f be entire s.t. $f(\mathbb{R}) \subseteq \mathbb{R}$. Prove:

(i) $f^{(n)}(\mathbb{R}) \subseteq \mathbb{R}$, $\forall\, n \in \mathbb{N}$.

(ii) $f(\overline{z}) = \overline{f(z)}$, $\forall\, z \in \mathbb{C}$.

[Hint: Find Taylor's expansion of f at 0.]