

$$\begin{aligned}
 \gamma'(t) \overline{\gamma(t)} &= (-2\sin t + 3i\cos t)(2\cos t - 3i\sin t) \\
 &= -4\sin t \cos t + 6i\sin^2 t + 6i\cos^2 t + \\
 &\quad + 9\sin t \cos t \\
 &= 5\sin t \cos t + 6i
 \end{aligned}$$

$$\Rightarrow \int_{\gamma} \frac{dz}{z} = \int_0^{2\pi} \frac{5\sin t \cos t + 6i}{4\cos^2 t + 9\sin^2 t} dt =$$

$$= \underbrace{\quad}_{A} + 6i \left(\int_0^{2\pi} \frac{dt}{4\cos^2 t + 9\sin^2 t} \right) \rightarrow J$$

$$\Rightarrow 2+i = A + 6iJ$$

$$\Rightarrow J = 2\pi/6 = \pi/3$$

(11) (ii) Να βρείτε το ανάπτυγμα Taylor γύρω από το

$$z_0 = \pi, \text{ ως } f(z) = \cos^2 z.$$

Λύση: $f(z) = \frac{1 + \cos(2z)}{2} = \frac{1}{2} + \frac{1}{2} \cos(2z)$

$$w = z - \pi \Rightarrow z = \pi + w$$

$$\Rightarrow \cos(2z) = \cos(2\pi + 2w) = \cos(2w) =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2w)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} w^{2n} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} (z - \pi)^{2n} \quad \text{κ. λ. π.}$$

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$$f(z) = \frac{(1 - \cos z)^2}{(z - 1 - z) \sin^2 z}, \quad \lim_{z \rightarrow 0} f(z) = ?$$

Λύση:

$$f(z) = \frac{(z^2/2! - z^4/4! + z^6/6! - \dots)^2}{(z^2/2! + z^3/3! + z^4/4! + \dots)(z - z^3/3! + z^5/5! - \dots)^2}$$
$$= \frac{z^4 \left(\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \dots \right)^2}{z^4 \left(\frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \dots \right) \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots \right)^2}$$

$$\Rightarrow \lim_{z \rightarrow 0} f(z) = \frac{(1/2!)^2}{1/2! \cdot 1^2} = 1/2.$$

(16) Eow $f: \mathbb{C} \rightarrow \mathbb{C}$ ar kya $\mu \in f(\mathbb{R}) \subset \mathbb{R}$.

(i) $\forall n \in \mathbb{N}, f^{(n)}(\mathbb{R}) \subset \mathbb{R}$.

(ii) $f(\bar{z}) = \overline{f(z)}, \forall z \in \mathbb{C}$.

Proof:

(ii) $\forall z \in \mathbb{C}, f(z) = \sum_{n=0}^{\infty} \left[\frac{f^{(n)}(0)}{n!} \right] z^n$ (0. Taylor)

$$\begin{aligned} \Rightarrow \overline{f(z)} &= \sum_{n=0}^{\infty} \overline{\left(\frac{f^{(n)}(0)}{n!} \right)} z^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \bar{z}^n = f(\bar{z}). \end{aligned}$$

$\in \mathbb{R}$ \rightarrow μ \rightarrow $\mu \in \mathbb{R}$

(i) Θα δ.ο. $f'(\mathbb{R}) \subset \mathbb{R}$. Έστω $x_0 \in \mathbb{R}$.

Επειδή f διαφορ. στο x_0 , τότε

$$f'(x_0) = \lim_{z \rightarrow x_0} \frac{f(z) - f(x_0)}{z - x_0}$$

$$\Rightarrow f'(x_0) = \lim_{\substack{z \rightarrow x_0 \\ z \in \mathbb{R}}} \frac{f(z) - f(x_0)}{z - x_0} \in \mathbb{R}$$

αφαι $f(z) \in \mathbb{R}$, $\forall z \in \mathbb{R}$ \wedge $f(x_0) \in \mathbb{R}$.

Θα δείξουμε με επαγωγή στο n ότι:

11 Εάν $f: \mathbb{C} \rightarrow \mathbb{C}$ αλgebra με $f(\mathbb{R}) \subset \mathbb{R}$, ισχύει
 $f^{(n)}(\mathbb{R}) \subset \mathbb{R}$.

• Για $n=0$, ισχύει.

• Υποθέτουμε ότι ισχύει για κάποιο $n \in \mathbb{N}$.

Έστω $f: \mathbb{C} \rightarrow \mathbb{C}$ αλgebra με $f(\mathbb{R}) \subset \mathbb{R}$.

Τότε, $f'(\mathbb{R}) \subset \mathbb{R}$. Εφαρμόζοντας την

επαγωγή υποθέτουμε για " f " = f' και για n ,

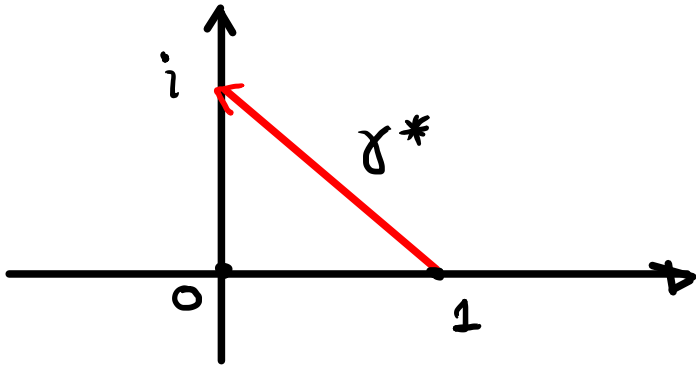
ναίμων
Αρα, $(f')^{(n)}(\mathbb{R}) \subset \mathbb{R} \Rightarrow f^{(n+1)}(\mathbb{R}) \subset \mathbb{R}$.
ισχύει και για $n+1$.

ΦΥΛΛΑΔΙΟ 2

① (iv)

$$\int_{\gamma} \frac{\text{Log } z}{z} dz = ? , \quad \gamma = [1, i]$$

Λύση: $\gamma^* \subset U = \mathbb{C} \setminus (-\infty, 0]$



ε'η $F(z) = \frac{1}{2} \text{Log}^2 z$

είναι ορίζομενη σε U

$\mu \in F'(z) = \frac{\text{Log } z}{z}, \quad \forall z \in U$

$$\begin{aligned} \Rightarrow \int_{\gamma} \frac{\text{Log } z}{z} dz &= F(i) - F(1) \\ &= \frac{1}{2} \text{Log}^2 i = \frac{1}{2} (\ln|i| + i \text{Arg}(i))^2 \end{aligned}$$

$$= \frac{1}{2} \left(\frac{\pi i}{2} \right)^2 = -\frac{\pi^2}{8}.$$

(vi) $\int_{\gamma} \overline{z^2} e^z dz = ?$, $\gamma(t) = e^{it}$, $t \in [0, \pi]$.

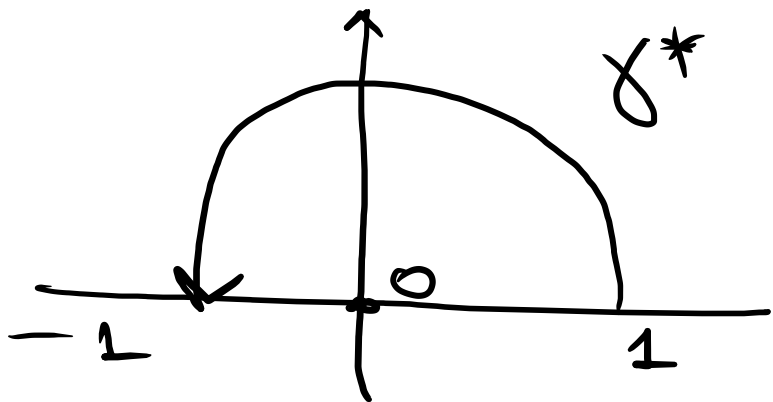
$$\forall z \in \gamma^*, |z| = 1 \Rightarrow \bar{z} = 1/z$$

$$\Rightarrow \overline{z^2 e^z} = \bar{z}^2 e^{\bar{z}} = \frac{1}{z^2} e^{1/z} = -\left(e^{1/z}\right)'$$

$$F(z) = -e^{1/z}, \text{ so } F'(z) = \frac{1}{z^2} e^{1/z}, \forall z \in U.$$

$$U = \mathbb{C} \setminus \{z\} \supset \gamma^*$$

$$\Rightarrow \int_{\gamma} \overline{z^2 e^z} dz = \int_{\gamma} \frac{1}{z^2} e^{1/z} dz = -e^{1/z} \Big|_{z=1}^{z=-1}$$



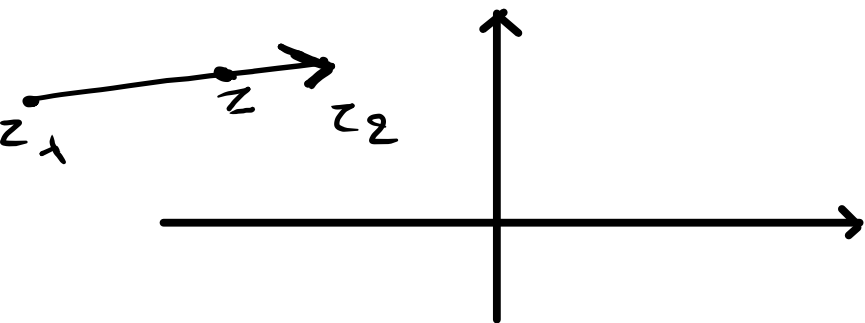
$$= -(e^{-1} - e)$$

$$= e - e^{-1}$$

(5) Für $z_1, z_2 \in \mathbb{C}$ mit $\underline{\operatorname{Re}(z_1)} \leq 0$, $\underline{\operatorname{Re}(z_2)} \leq 0$,
 valid. $|e^{z_1} - e^{z_2}| \leq |z_1 - z_2|$.

Λ Lösung:

$$\int_{[z_1, z_2]} e^z dz = \int_{[z_1, z_2]} (e^z)' dz = e^z \Big|_{z_1}^{z_2} = e^{z_2} - e^{z_1} \quad (*)$$



$\forall z \in [z_1, z_2], \operatorname{Re}(z) \leq 0$

$$\Rightarrow |e^z| = e^{\operatorname{Re}(z)} \leq 1$$

$$\text{ML-avio.} \Rightarrow \left| \int_{[z_1, z_2]} e^z dz \right| \leq 1 \cdot |z_1 - z_2| = |z_1 - z_2|$$

$$\Rightarrow |e^{z_2} - e^{z_1}| \leq |z_1 - z_2|.$$

$$(*) \quad \forall z \in [z_1, z_2] \text{ exists } z = (1-t)z_1 + tz_2 \quad \text{for all } t \in [0, 1]$$

$$\begin{aligned} \Rightarrow \operatorname{Re}(z) &= \operatorname{Re}[(1-t)z_1] + \operatorname{Re}(tz_2) \\ &= (1-t)\operatorname{Re}z_1 + t\operatorname{Re}z_2 \leq 0 + 0 = 0. \end{aligned}$$

8 (i) Έάν $\varphi: [a, b] \rightarrow \mathbb{C}$ διαφορ. να δ.ο.

$$\frac{d}{dt} [|\varphi(t)|^2] = 2 \operatorname{Re}[\varphi'(t) \overline{\varphi(t)}], \forall t \in [a, b].$$

Λύση: (i)

$$\varphi(t) = \varphi_1(t) + i\varphi_2(t), \quad t \in [a, b],$$

με $\varphi_1, \varphi_2: [a, b] \rightarrow \mathbb{R}$ διαφορίσιμες.

$$|\varphi(t)|^2 = \varphi_1(t)^2 + \varphi_2(t)^2$$

$$\Rightarrow \frac{d}{dt} [|\varphi(t)|^2] = 2\varphi_1(t)\varphi_1'(t) + 2\varphi_2(t)\varphi_2'(t).$$

Σημείωση,

$$\varphi'(t)\overline{\varphi(t)} = [\varphi_1'(t) + i\varphi_2'(t)] \cdot [\varphi_1(t) - i\varphi_2(t)]$$

$$= [(\varphi_1'\varphi_1 + \varphi_2\varphi_2') + i(\varphi_2'\varphi_1 - \varphi_1'\varphi_2)](t),$$

$\forall t \in (a, b)$

$$\Rightarrow 2\operatorname{Re} [\varphi'(t)\overline{\varphi(t)}] = 2(\varphi_1'\varphi_1 + \varphi_2\varphi_2')(t) = \frac{d}{dt} [|\varphi(t)|^2].$$

(ii) Έστω $f: U \rightarrow \mathbb{C}$ ολόμορφη (Holomorphic) κ' γ απλή, κατ'εξ'αίρεση
άρα με $\gamma^* \subset U$. Να δ-ο. $\int_{\gamma} f'(z)\overline{f(z)} dz$ φανερ.

Lemma: Es sei $\gamma: [a, b] \rightarrow \mathbb{C}$ mit $a, b \in \mathbb{R}$.

$$\int_{\gamma} f'(z) \overline{f(z)} dz = \int_a^b \underline{f'(\gamma(t))} \overline{f(\gamma(t))} \underline{\gamma'(t)} dt$$

$$= \int_a^b \frac{d}{dt} [f(\gamma(t))] \cdot \overline{f(\gamma(t))} dt \quad \underline{\varphi(t) = f(\gamma(t))}$$

$$= \int_a^b \varphi'(t) \overline{\varphi(t)} dt$$

$$\Rightarrow \operatorname{Re} \left(\int_{\gamma} f' \overline{f} \right) = \int_a^b \operatorname{Re} [\varphi'(t) \overline{\varphi(t)}] dt \quad \underline{(i)}$$

$$= \frac{1}{2} \int_a^b \frac{d}{dt} [|\varphi(t)|^2] dt =$$

$$= \frac{1}{2} (|\varphi(b)|^2 - |\varphi(a)|^2)$$

$$= \frac{1}{2} (|f(\gamma(b))|^2 - |f(\gamma(a))|^2) \quad \begin{array}{l} \gamma \text{ ungerade} \\ \underline{\underline{(\gamma(a) = \gamma(b))!}} \end{array}$$

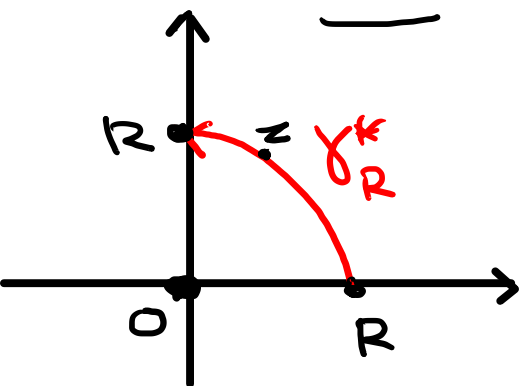
$= 0 !!$

④ Na s.o.

$$\lim_{R \rightarrow +\infty} \int_{\gamma_R} \frac{e^{iz}}{1+z^2} dz = 0, \quad 0 < \pi < 2\pi$$

$$\gamma_R(t) = Re^{it}, \quad t \in [0, \pi/2], \quad R > 0.$$

$\wedge \gamma \Sigma_H$



$$\text{z seen } z = x + iy \in \gamma_R^*$$

$$\Rightarrow |z| = R \quad \text{is}$$

$$iz^2 = i(x^2 - y^2 + 2ixy)$$

$$= -2xy + i(x^2 - y^2)$$

$$\Rightarrow |e^{iz^2}| = e^{\operatorname{Re}(iz^2)} = e^{-2xy} \leq 1$$

Εμπλοήον, $\forall z \in \gamma_R^*$,

$$|z^2 + 1| >, |z|^2 - 1 = R^2 - 1$$

$$\implies \forall R > 1, \forall z \in \gamma_R^*, \quad \frac{1}{|z^2 + 1|} \leq \frac{1}{R^2 - 1}$$

$$\implies \left| \frac{e^{iz^2}}{1+z^2} \right| \leq \frac{1}{R^2 - 1} \quad \xrightarrow{ML} \left| \int_{\gamma_R} \frac{e^{iz^2}}{1+z^2} dz \right| \leq$$

$$\leq \frac{1}{R^2 - 1} \frac{\pi R}{2} \quad \begin{matrix} R \rightarrow +\infty \\ \downarrow \\ 0 \end{matrix}$$

$$\textcircled{7} \int_{\gamma_r} \operatorname{Re} z \, dz = ? , \quad \gamma_r(t) = re^{it}, \quad t \in [0, 2\pi] \quad (r > 0).$$

Να δ.ο. η $\operatorname{Re} z \, dz$ έχει παράγωγο σε κανένα ανοικτό $U \subset \mathbb{C}$ με $0 \in U$.

Λύση:

$$\int_{\gamma_r} \operatorname{Re} z \, dz = \int_0^{2\pi} \operatorname{Re}(re^{it}) i re^{it} \, dt =$$

$$= \int_0^{2\pi} (r \cos t) i r (\cos t + i \sin t) \, dt$$

$$= ir^2 \left(\int_0^{2\pi} \cos^2 t \, dt + i \int_0^{2\pi} \underbrace{\cos t \sin t}_{\leftarrow 0} \, dt \right)$$

$$= ir^2 \int_0^{2\pi} \frac{1 + \cos(rt)}{2} dt =$$

$$= i\pi r^2 \neq 0$$

Εστω U ανοικτός \mathbb{C} . Υποθέτουμε ότι η $f(z) = \operatorname{Re} z$
έχει παράγωγο στο U . $\exists r > 0 \mid \gamma_r^* \subset U$ όπου
 $\gamma_r(t) = re^{it}$. Τότε, γ_r κατ'εξοχή $\Rightarrow \int_{\gamma_r} f(z) dz = 0$
 (Απόδειξη!)