ΦΥΛΛΑΔΙΟ 3

(5) $z_0 \in \mathbb{C}$, $\operatorname{Im} z_0 < 0$, $R > 0$, $\gamma_R(t) = Re^{it}$, $t \in [\pi, 2\pi]$.

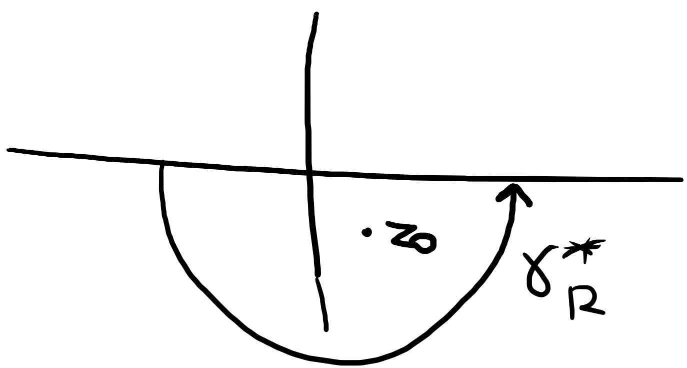
(i) Να δ.ο.

$$\lim_{R\to+\infty} \int_{\gamma_R} \frac{dz}{z(z-z_0)} = 0, \quad \lim_{R\to+\infty} \int_{\gamma_R} \frac{dz}{z-z_0} = \pi i.$$

Λύση: $\forall z \in \gamma_R^*$, $\quad |z(z-z_0)| = |z|\cdot|z-z_0|$

$$\geq R\,(R-|z_0|)$$

για $R > |z_0|$

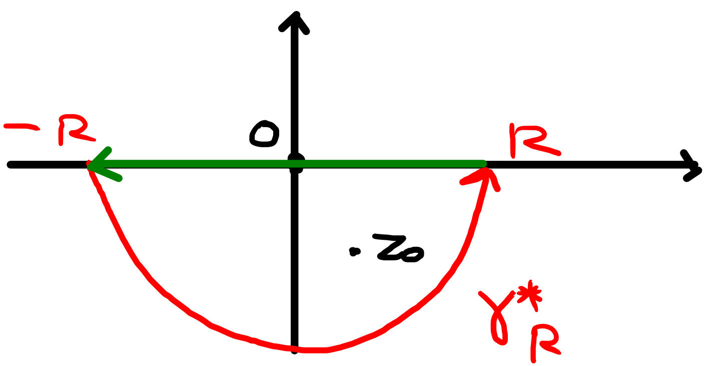


$$\Rightarrow \left|\frac{1}{z(z-z_0)}\right| \leq \frac{1}{R(R-|z_0|)}$$

ML anor.

$$\Longrightarrow \forall R > |z_0| \quad \left| \int_{\gamma_R} \frac{dz}{z(z-z_0)} \right| \le \pi R \, \frac{1}{\cancel{R}\,(R-|z_0|)} \xrightarrow{R\to+\infty} 0$$

$$\int_{\gamma_R} \frac{dz}{z(z-z_0)} = \int_{\gamma_R} \frac{dz}{z-z_0} - \int_{\gamma_R} \frac{dz}{z}$$



$$\int_{\gamma_R} \frac{dz}{z} = \int_{\pi}^{2\pi} \frac{iRe^{it}}{Re^{it}}\,dt$$

$$= \pi i$$

$$\Longrightarrow \lim_{R\to+\infty} \int_{\gamma_R} \frac{dz}{z-z_0} = \pi i$$

(ii)

$$\lim_{R\to+\infty}\int_{-R}^{R}\frac{dt}{t-z_0}=\,?$$

Θεωρούμε την $\Gamma_R=\gamma_R+[R,-R]$ = κλειστή τμ. λεία απλή

$$\xRightarrow[\text{Cauchy}]{\text{Ο.Τ.}}\quad \int_{\Gamma_R}\frac{dz}{z-z_0}=2\pi i\cdot 1=2\pi i$$

$$\int_{\gamma_R}\frac{dz}{z-z_0}-\int_{-R}^{R}\frac{dt}{t-z_0}=2\pi i$$

$$\Big\downarrow{\scriptstyle R\to+\infty}$$

$$\pi i \quad (\text{βλ. (i)})$$

$$\Longrightarrow \lim_{R\to+\infty}\int_{-R}^{R}\frac{dt}{t-z_0}=-\pi i$$

(7) Έστω $f: U \to \mathbb{C}$ ολόμορφη, όπου U ανοικτό με $D = \{ z : |z| \leq 1 \} \subset U$ κ' $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$.

Να δ.ο. $\displaystyle\int_\gamma \overline{f(z)}\, dz = 2\pi i\, \overline{f'(0)}$.

<u>Απόδ:</u> $\displaystyle\int_\gamma \overline{f(z)}\, dz = \int_0^{2\pi} \overline{f(e^{it})}\, i e^{it}\, dt =$

$\displaystyle = i \int_0^{2\pi} \overline{e^{-it} f(e^{it})}\, dt = i\, \overline{\int_0^{2\pi} e^{-it} f(e^{it})\, dt}$

Αλλά $\displaystyle\int_0^{2\pi} e^{-it} f(e^{it})\, dt = \frac{1}{i} \int_0^{2\pi} \frac{f(e^{it})}{(e^{it})^2}\, i e^{it}\, dt$

$\displaystyle = \frac{1}{i} \int_\gamma \frac{f(z)}{z^2}\, dz = \frac{1}{i}\, 2\pi i\, f'(0) = 2\pi f'(0)$

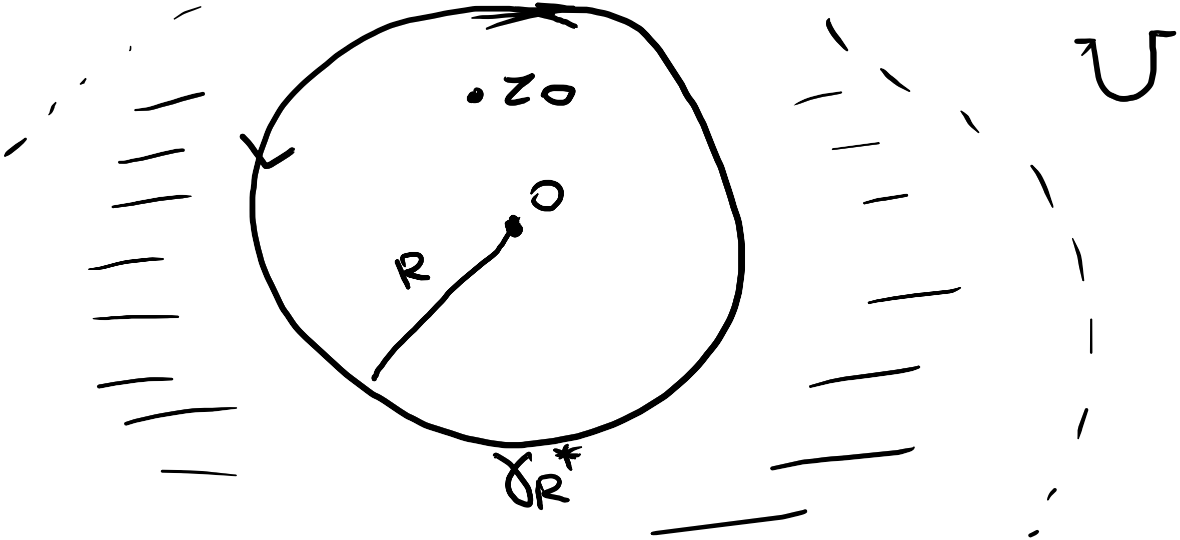
$$\Longrightarrow \int_\gamma \overline{f(z)}\,dz = i\,\overline{2\pi f'(0)} = 2\pi i\,\overline{f'(0)}.$$

$\boxtimes$

(8) Έστω f ολόμορφη σε ανοικτό $U \supset D[0,R] = \{z \in \mathbb{C} : |z| \leq R\}$ $(R>0)$. Εάν $f(z_0)=0$ για κάποιο z_0 με $|z_0|<R$, να δ.ο.

$$|f(0)| \leq M_R \frac{|z_0|}{R-|z_0|}, \text{ όπου}$$

$$M_R = \max\{|f(z)| : |z| = R\}.$$



$\gamma_R(t) = Re^{it}$,
$t \in [0, 2\pi]$.

O.T. Cauchy $\Longrightarrow$

$$f(0) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{z} dz, \quad \underset{\underset{0}{\shortparallel}}{f(z_0)} = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{z - z_0} dz$$

$\Longrightarrow$

$$\Rightarrow \quad |f(0)| = |f(0) - f(z_0)| =$$

$$= \frac{1}{2\pi} \left| \int_{\gamma_R} \left[\frac{f(z)}{z} - \frac{f(z)}{z - z_0} \right] dz \right|$$

$$= \frac{1}{2\pi} \left| \int_{\gamma_R} f(z) \left(\frac{1}{z} - \frac{1}{z - z_0} \right) dz \right|$$

$$= \frac{1}{2\pi} \left| -z_0 \int_{\gamma_R} \frac{f(z)}{z(z - z_0)} \, dz \right|$$

$$= \frac{|z_0|}{2\pi} \left| \int_{\gamma_R} \frac{f(z)}{z(z - z_0)} dz \right| .$$

Αλλά, $\underline{\forall z \in \gamma_R^*}$, $|f(z)| \leq M_R$ και

$$|z(z-z_0)| = R\cdot|z-z_0| \geq R(|z|-|z_0|) = R(R-|z_0|)$$

$$\Longrightarrow \left|\frac{f(z)}{z(z-z_0)}\right| \leq \frac{M_R}{R(R-|z_0|)}$$

ML ανισ.

$$\Longrightarrow \left|\int_{\gamma_R} \frac{f(z)}{z(z-z_0)}\,dz\right| \leq 2\pi \cancel{R}\frac{M_R}{\cancel{R}(R-|z_0|)} = 2\pi\frac{M_R}{R-|z_0|}$$

$$\Longrightarrow |f(0)| \leq \frac{|z_0|}{\cancel{2\pi}}\cancel{2\pi}\frac{M_R}{R-|z_0|} = \cdots \checkmark \quad \boxtimes$$

(10) Έστω $P(z)$ πολυώνυμο βαθμού $n \geq 2$ με μεγίστου βαθμού όρο $a_n z^n$ ($a_n \in \mathbb{C}$, $a_n \neq 0$).

(i) Να δ.ο.

$$\lim_{|z|\to\infty} \left| \frac{P(z)}{a_n z^n} - 1 \right| = 0$$

κ' να συμπεραίνετε ότι $\exists\, R_0 > 0 \,|$ $\forall z \in \mathbb{C}$ με $|z| > R_0$, ισχύει

$$|P(z)| > \frac{|a_n|}{2} \cdot |z|^n$$

Απόδ:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

$$= a_n z^n + \sum_{k=0}^{n-1} a_k z^k$$

$$\Rightarrow \left| \frac{P(z)}{a_n z^n} - 1 \right| = \left| \sum_{k=0}^{n-1} \frac{a_k}{a_n} \frac{1}{z^{n-k}} \right|$$

$$\leq \sum_{k=0}^{n-1} \frac{|a_k|}{|a_n|} \left(\frac{1}{|z|} \right)^{n-k} \xrightarrow{|z| \to \infty} 0$$

$$\Rightarrow \lim_{|z| \to \infty} \left| \frac{P(z)}{a_n z^n} - 1 \right| = 0.$$

Για $\varepsilon = 1/2$, $\exists R_0 > 0 \mid$ εάν $|z| > R_0$, ισχύει

$$\left| \frac{P(z)}{a_n z^n} - 1 \right| < 1/2$$

Αλλά $\left| \frac{P(z)}{a_n z^n} - 1 \right| \geq 1 - \left| \frac{P(z)}{a_n z^n} \right|$

$$\Rightarrow \quad 1 - \frac{|P(z)|}{|a_n| \cdot |z|^n} < 1/2$$

$$\Rightarrow \quad 1/2 < \frac{|P(z)|}{|a_n| \cdot |z|^n} \Rightarrow |P(z)| > \underbrace{\frac{1}{2}|a_n|}\cdot |z|^n$$

$$\forall\ |z| > R_0$$

(ii) $\lim\limits_{R\to+\infty} \int_{\gamma_R} \frac{dz}{P(z)} = 0$, όπου

$\gamma_R(t) = Re^{it}$, $t \in [0, 2\pi]$ $(R > 0)$.

Αποδ: $\forall z \in \gamma_R^*$, $|P(z)| \overset{(i)}{>} \frac{1}{2}|a_n| \cdot |z|^n$

$= \frac{1}{2}|a_n| \cdot R^n$

$\Longrightarrow \left|\frac{1}{P(z)}\right| < \frac{2}{|a_n|} \cdot \frac{1}{R^n} \overset{\text{ML-ανισ.}}{\Longrightarrow}$

$\Longrightarrow \left|\int_{\gamma_R} \frac{dz}{P(z)}\right| \leq \frac{2}{|a_n|} \cdot \frac{1}{R^n} 2\pi R = \frac{4\pi}{|a_n|} \cdot \left(\frac{1}{R}\right)^{n-1}$

$$\overset{n\geq 2}{\Longrightarrow} \quad \lim_{R\to+\infty} \left| \int_{\gamma_R} \frac{dz}{P(z)} \right| = 0.$$

(iii) Έστω γ απλή κλειστή τμ. λεία καμπύλη που περικλείει όλες τις ρίζες του $P(z)$.

Να δ.ο. $\displaystyle\int_\gamma \frac{1}{P(z)}\,dz = 0.$

Λύση:

$\exists R_1 > 0 \mid$

$\gamma^* \subset \operatorname{int} \gamma^*_{R_1}$

με $\gamma_{R_1}(t) = R_1 e^{it}$, $t \in [0, 2\pi]$

$\forall R > R_1$, $\gamma^* \subset \operatorname{int} \gamma^*_{R_1} \subset \operatorname{int} \gamma^*_R$

κ' το πεδίο $\underbrace{\qquad}_{U}$ μεταξύ των γ^*, γ^*_R δεν περιέχει ρίζες του $P(z)$ δηλ. δεν περιέχει ανώμαλα της $1/P$

$\Longrightarrow$ η $\frac{1}{P}$ είναι ολόμορφη στο U $\xRightarrow{\text{Αρχή Παραμ.}}$

$$\Longrightarrow \int_\gamma \frac{dz}{P(z)} = \int_{\gamma_R} \frac{dz}{P(z)}, \quad \forall R > R_1$$

Αλλά β' μέλος $\xrightarrow{R\to+\infty} 0$ (βλ. (ii)) $\Longrightarrow$

$$\Rightarrow \int_\gamma \frac{dz}{P(z)} = 0.$$

$\boxtimes$

13 Να βρείτε τη σειρά Taylor της $f(z) = \frac{z^5}{1+z^4}$ γύρω από το 0 κ' $f^{(21)}(0) = ?$

<u>Λύση:</u> Θέτω $w = -z^4$

$$\frac{1}{1+z^4} = \frac{1}{1-w} \underset{=}{\overset{|w|<1}{=}} \sum_{n=0}^{\infty} w^n =$$

$$= \sum_{n=0}^{\infty} (-1)^n z^{4n} \quad \text{για } \underline{|z|<1}$$

$$\Rightarrow \quad f(z) = z^5 \cdot \sum_{n=0}^{\infty} (-1)^n z^{4n} = \sum_{n=0}^{\infty} (-1)^n z^{4n+5},$$

για $|z|<1$.

$\forall k \in \mathbb{N}$, $$\frac{f^{(k)}(0)}{k!} = \text{συντελ.}\,(z^k)$$

$$4n+5 = 21 \Rightarrow 4n = 16 \Rightarrow \underline{n=4}$$

$$\Rightarrow \frac{f^{(21)}(0)}{21!} = \text{συντελ}(z^{21}) = (-1)^4 = 1$$

$$\Rightarrow \underline{f^{(21)}(0) = 21!}$$

(14) Για $|z|<1$, $\sum_{n=1}^{\infty} n^2 z^n = ?$

__Λύση:__ $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad , \quad |z|<1$

(παραγωγίζω)

$\Rightarrow \quad \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1}$

$\Rightarrow \quad \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n$

$\Rightarrow \quad \left[\frac{z}{(1-z)^2}\right]' = \sum_{n=1}^{\infty} n^2 z^{n-1}$

$$\Rightarrow \quad z\left[\frac{z}{(1-z)^2}\right]' = \sum_{n=1}^{\infty} n^2 z^n,$$

κ.λ.π.

ΦΥΛΛΑΔΙΟ 1

12 (i) Εάν $x_0<0$, να δ.ο. $\lim_{w\to x_0} \operatorname{Log} w$ δεν υπάρχει.

Λύση: $\quad z_n = |x_0|\, e^{i(\pi-\frac{1}{n})}, \quad w_n = |x_0|\, e^{i(-\pi+\frac{1}{n})}$

$$z_n \xrightarrow{n} |x_0|\cdot e^{i\pi} = -|x_0| = x_0,$$

όμοια

$$w_n \to x_0$$

$$\mathrm{Log}\, z_n = \ln|x_0| + i\left(\pi - \frac{1}{n}\right) \xrightarrow{n} \ln|x_0| + i\pi$$

$$\neq$$

$$\mathrm{Log}\, w_n = \ln|x_0| + i\left(-\pi + \frac{1}{n}\right) \xrightarrow{n} \ln|x_0| - i\pi$$

$\Longrightarrow$ $\nexists$ το $\displaystyle\lim_{w \to x_0} \mathrm{Log}\, w$.

<u>(ii)</u> $u(x,y) = \frac{1}{2}\ln(x^2+y^2) = \ln\sqrt{x^2+y^2}$

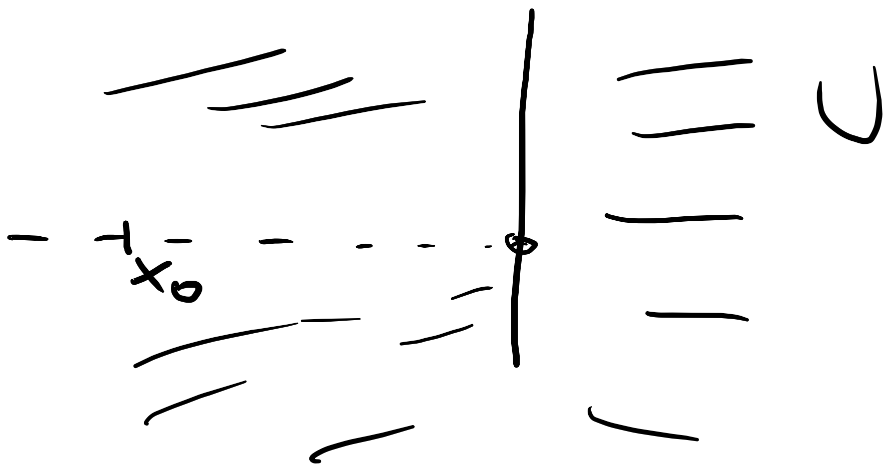
$= \ln|x+iy|$

Εάν $U = \mathbb{C} \setminus (-\infty, 0]$, τότε

$F = \mathrm{Log}$ ολόμορφη στο U κ' $u = \mathrm{Re}\, F$.

Ερώτημα: $\exists\ f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ ολόμορφη με $\operatorname{Re} f = u$?

<u>Όχι</u>



Έστω ότι $\exists\ f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ ολόμορφη με $\operatorname{Re} f = u$.

Τότε, $\forall z \in U$, $\operatorname{Re} f(z) = u(z) = \operatorname{Re} F(z)$

$\Longrightarrow \operatorname{Re}\left[F(z) - f(z)\right] = 0, \ \forall z \in U =$ πεδίο

U πεδίο

$\Longrightarrow$ $\underline{F(z) = f(z) + c, \quad \forall z \in U.} \ (*)$

Επιλέγω $x_0 < 0$.

$\Longrightarrow$ f συνεχής στο x_0 $\Longrightarrow$ $\lim\limits_{z \to x_0} f(z) = f(x_0)$

$\overset{(*)}{\Longrightarrow}$ $\lim\limits_{z \to x_0} \operatorname{Log} z = f(x_0) + c$ (Άτοπο!)

λόγω του (i).