EXERCISES 5.

Exercise 1: Show that $C[0,1]$ is w^*-dense in $L^\infty[0,1]$.

Proof: Suppose not. Then we can find $g \in L^\infty[0,1] \setminus \overline{C[0,1]}^{w^*}$. By the strong separation theorem we can find $f \in \left(L^\infty[0,1]_{w^*}\right)^* = L^1[0,1]$ $f \neq 0$ and $\varepsilon > 0$ s.t

$$\langle f, g\rangle - \varepsilon \geq \langle f, u\rangle \quad \forall\, u \in C[0,1]$$

Recall $\langle f, h\rangle = \int_0^1 f h \, dt \quad \forall\, f \in L^1[0,1],\ h \in L^\infty[0,1]$.

Note that $\langle f, u\rangle = 0 \quad \forall\, u \in C[0,1]$

Let $t \in (0,1]$ and $\delta > 0$ small s.t $t - \delta > \delta$. We can find $u \in C[0,1]$ with values in $[0,1]$ s.t

$$u\Big|_{[\delta, t-\delta]} \equiv 1, \quad u\Big|_{(t,1]} \equiv 0 \qquad \left(u(s) = \frac{d(s,C)}{d(s,C) + d(s,C^c)}\right).$$

We have

$$\left|\int_0^t f(s)\,ds\right| = \left|\int_0^t f(s)\,ds - \int_0^1 f(s)u(s)\,ds\right|$$

$$= \left|\int_0^t f(s)\,ds - \int_0^t f(s)u(s)\,ds\right|$$

$$= \left|\int_0^t f(s)(1 - u(s))\,ds\right|$$

$$\leq \int_0^\delta |f(s)|\,ds + \int_{t-\delta}^\delta |f(s)|\,ds$$

Let $\delta \downarrow 0$ to conclude that

$$\int_0^t f(s)\,ds = 0 \quad \forall\, t \in [0,1]$$

$$\Rightarrow f \equiv 0 \text{, a contradiction}$$

QED

<u>Exercise 2:</u> Suppose $\{f_n\}_{n\in\mathbb{N}} \subseteq L^1(\Omega)$ $f_n \geq 0$, $f_n \xrightarrow{a.e} f$ and

$$\int_\Omega f_n\,dx \longrightarrow \int_\Omega f\,dx \quad \text{with } f \in L^1(\Omega),\ f \geq 0.$$

Show that $\forall A \subseteq \Omega$ measurable, we have

$$\int_A f_n\,dx \to \int_A f\,dx.$$

<u>Proof:</u> Let $h_n = f - f_n$. Then

$$h_n \xrightarrow{a.e} 0,$$

$$\Rightarrow h_n^+, h_n^- \xrightarrow{a.e} 0.$$

We have $0 \leq h_n^+ \leq f$ $\forall n \in \mathbb{N}$. By the DCThm we have

$$\int_\Omega h_n^+\,dx \longrightarrow 0.$$

Also by hypothesis

$$\int_\Omega h_n\,dx \longrightarrow 0.$$

Therefore

$$\int_\Omega h_n^-\,dx \to 0,$$

$$\Rightarrow \|f - f_n\|_1 = \int_\Omega h_n^+\,dx + \int_\Omega h_n^-\,dx \longrightarrow 0,$$

$$\Rightarrow f_n \longrightarrow f \text{ in } L^1(\Omega),$$

$$\Rightarrow \{|f_n|\}_{n\in\mathbb{N}} \text{ uniformly integrable.}$$

Suppose $A \subseteq \Omega$ measurable. Then

$$\chi_A |f_n - f| \leqslant |f_n| + |f| \quad \forall n \in \mathbb{N},$$

$$\Rightarrow \chi_A f_n \xrightarrow{a.e} \chi_A f.$$

By Vitali's Thm. we have

$$\int_A f_n dx = \int_\Omega \chi_A f_n dx \longrightarrow \int_\Omega \chi_A f dx = \int_A f dx$$

QED

<u>Exercise 3:</u> Let $1<p<\infty$ and $\{f_n\}_{n\in\mathbb{N}} \subseteq L^p[0,1]$ a bounded sequence. Assume $f_n \xrightarrow{a.e} f$. Show that $f_n \xrightarrow{w} f$ in $L^p[0,1]$

<u>Proof:</u> We may assume that

$$f_n \xrightarrow{w} \hat{f} \text{ in } L^p[0,1],$$

$$\Rightarrow \langle f_n, h\rangle \longrightarrow \langle f, h\rangle \quad \forall\ h \in L^{p'}[0,1],$$

$$\Rightarrow \int_0^1 f_n h\, dx \longrightarrow \int_0^1 \hat{f} h\, dx$$

But $\{|f_n|\}_{n\in\mathbb{N}}$ unif. integrable, hence

$$\{f_n h\}_{n\in\mathbb{N}} \subseteq L^1(\Omega) \text{ unif. integrable}$$

$$f_n h \xrightarrow{a.e} f h.$$

So, by Vitali's thm

$$\int_0^1 f_n h\,dx \longrightarrow \int_0^1 f h\,dx,$$

$$\Rightarrow \int_0^1 [f-\hat{f}]\,h\,dx = 0 \quad \forall\, h \in L^{p'}[0,1],$$

$$\Rightarrow \quad f = \hat{f} \ \text{a.e.}$$

QED

Exercise 4: Produce a sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq L^p[0,1]$ $(1 \leq p \leq \infty)$ s.t

$$f_n \xrightarrow{a.e} 0, \quad \|f_n\|_p \longrightarrow \infty$$

Proof: Consider the sequence

$$f_n(x) = \begin{cases} e^n & \text{if } x \in [0, \frac{1}{n}] \\ 0 & \text{if } x \in (\frac{1}{n}, 1]. \end{cases}$$

Then $f_n \xrightarrow{a.e} 0$ but

$$\|f_n\|_p = \frac{1}{n} e^{np} \longrightarrow +\infty \quad (1 \leq p < \infty)$$

$$\|f_n\|_\infty = e^n \longrightarrow +\infty \quad (p = \infty).$$

QED

Fact: If $1 \leq p < q \leq \infty$

then $\ell^p \hookrightarrow \ell^q$ continuously.

Exercise 5: Show that $\ell^p \hookrightarrow \ell^q$ is not compact

Proof: Arguing by contradiction suppose that the embedding is compact. Let $\{e_n\}_{n\in\mathbb{N}}$ the standard basis of ℓ^p

$$\|e_n\|_p = 1 \quad \forall n \in \mathbb{N}.$$

So, we can find a subsequence $\{e_{n_k}\}_{k\in\mathbb{N}} \subseteq \{e_n\}_{n\in\mathbb{N}}$ such that

$$e_{n_k} \longrightarrow u \text{ in } \ell^q \qquad u = \{u_m\}_{m\in\mathbb{N}}$$

Note that $u_m = 0 \quad \forall m \in \mathbb{N}$, hence $u = 0$. But

$\|e_n\|_q = 1 \quad \forall n \in \mathbb{N}$, a contradiction.

QED

Exercise 6: Let X, Y be Banach spaces. Show that

$$A \in \mathcal{L}_c(X,Y) \Leftrightarrow A^* \in \mathcal{L}_c(Y^*, X^*)$$

Proof: $\Rightarrow$ Schauder Theorem.

$\Leftarrow$ From the first part we have

$$A^{**} \in \mathcal{L}_c(X^{**}, Y^{**})$$

Let $i_X : X \longrightarrow X^{**}$ and $i_Y : Y \longrightarrow Y^{**}$ be the canonical embeddings. We have

$$A = i_Y \circ A^{**} \circ i_X \in \mathcal{L}_c(X,Y)$$

QED

Exercise 7: Let X be a Banach space, $C \subseteq X^*$ nonempty, w^*-closed. Show that given any $x^* \in X^*$ we can find $u^* \in C$ s.t

$$\|x^* - u^*\|_* = d(x^*, C).$$

Proof: We consider a minimizing sequence $\{u_n^*\}_{n\in\mathbb{N}} \subseteq C$, that is,

$$\|x^* - u_n^*\|_* \downarrow d(x^*, C)$$

Evidently $\{u_n^*\}_{n\in\mathbb{N}} \subseteq C$ is bounded and so by Alaoglu's theorem it is relatively w^*-compact. So, we can find a subnet $\{u_\alpha^*\}_{\alpha\in J}$ s.t.

$$u_\alpha^* \xrightarrow{w^*} u^* \text{ in } X^*.$$

Since C is w^*-closed, we have $u^* \in C$. Recall that the norm $\|\cdot\|_*$ of X^* is w^*-lower semicontinuous. So, we have

$$\|x^* - u^*\|_* \leq \liminf_{\alpha\in J} \|x^* - u_\alpha^*\|_* = d(x^*, C), \quad u^* \in C$$

$$\Rightarrow \|x^* - u^*\|_* = d(x^*, C).$$

QED

Exercise 8: Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $1 \leq p < \infty$. Show that $L^1(\Omega) \cap L^\infty(\Omega)$ is dense in $L^p(\Omega)$.

Proof: Using the interpolation inequality we have

$$L^1(\Omega) \cap L^\infty(\Omega) \subseteq L^p(\Omega).$$

[Recall the interpolation inequality

<u>If</u> $1 \leq p \leq q \leq \infty$ and $f \in L^p \cap L^q$

<u>then</u> $f \in L^r$ for all $p \leq r \leq q$ and

$$\|f\|_r \leq \|f\|_p^t \, \|f\|_q^{1-t}$$

with $\frac{1}{r} = \frac{t}{p} + \frac{1-t}{q}$]

Let $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$ with $|\Omega_n|_N < \infty$ and $u \in L^p(\Omega)$

Consider the truncation map $\tau_n : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\tau_n(s) = \begin{cases} s & \text{if } |s| \leq n \\ \frac{ns}{|s|} & \text{if } n < |s| \end{cases}$$

We set $f_n = \chi_{\Omega_n} (\tau_n \circ f) \quad \forall n \geq 1$

Then $f_n \in L^1 \cap L^\infty \quad \forall n \geq 1$ and $f_n \to f$ in $L^p(\Omega)$.

QED

Exercise 9: Suppose X, Y are Banach space and $A : X \to Y$ a linear operator such that $\forall y^* \in Y^*$ $y^* \circ A \in X^*$. Show that

$$A \in \mathcal{L}(X, Y).$$

<u>Proof:</u> Let $\{x_\alpha\}_{\alpha \in J} \subseteq X$ be a net s.t $x_\alpha \xrightarrow{w} x$ in X. Then

$$(y^* \circ A)(x_\alpha) \longrightarrow (y^* \circ A)(x),$$

$$\Rightarrow \langle y^*, A(x_\alpha) \rangle \longrightarrow \langle y^*, A(x) \rangle \quad \forall y^* \in Y^*$$

$$\Rightarrow A(x_\alpha) \xrightarrow{w} A(x),$$

$$\Rightarrow A \in \mathcal{L}(X_w, Y_w),$$

$$\Rightarrow A \in \mathcal{L}(X, Y).$$

QED

Exercise 10: Let X be an infinite dimensional Banach space. Show that ∂B_1 is a dense G_δ-subset of $\overline{B}_1$ with the w-topology

Proof: We know that

$$\overline{\partial B_1}^{w} = \overline{B}_1$$

So, it is w-dense

Also let $G_n = \{ x \in \overline{B}_1 : \|x\| > 1 - \frac{1}{n} \}$. Recall that $\|\cdot\|$ is w-lower semicontinuous. Hence $\forall n \in \mathbb{N}$ G_n is w-open. We have

$$\bigcap_{n \in \mathbb{N}} G_n = \partial B_1 .$$

QED

Remark: $f : X \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is lsc.

iff

(a) $\operatorname{epi} f = \{ (x, \lambda) : f(x) \leq \lambda \}$ is closed

or

(b) $\forall \lambda \in \mathbb{R} : L_\lambda = \{ x \in X : f(x) \leq \lambda \}$ is closed

Exercise 11: Let X be a reflexive Banach space, Y a Banach space $A \in \mathcal{L}_c(X,Y)$. Show that $\exists\ u \in X, \|u\|_X \leq 1$ s.t

$$\|A\|_{\mathcal{L}} = \|A(u)\|_Y.$$

Proof: By definition we have

$$\|A\|_{\mathcal{L}} = \sup\left[\|A(x)\|_Y : \|x\|_X \leq 1\right]$$

Consider $\{x_n\}_{n\in\mathbb{N}} \subseteq \overline{B}_1^X$ such that

$$\|A(x_n)\|_Y \uparrow \|A\|_{\mathcal{L}}$$

Since $A \in \mathcal{L}_c(X,Y)$, we see that $\overline{\{A(x_n)\}_{n\in\mathbb{N}}}^{\|\cdot\|_Y} \subseteq Y$ is compact.

So, we may assume that

$$A(x_n) \longrightarrow y \quad \text{in } Y \quad (\|y\|_Y = \|A\|_{\mathcal{L}})$$

Also $\overline{B}_1^X$ is w-compact (since X is reflexive). So, we may assume that

$$x_n \xrightarrow{w} x \text{ in } X,$$

$$\Rightarrow A(x_n) \xrightarrow{w} A(x) \text{ in } Y \quad (\text{since } A \in \mathcal{L}(X_w, Y_w)),$$

$$\Rightarrow A(x) = y,$$

$$\Rightarrow \|A(x)\|_Y = \|A\|_{\mathcal{L}}.$$

QED

Exercise 12: Suppose $1<p<\infty$, $\{f_n, f\}_{n\in\mathbb{N}} \subseteq L^p[0,1]$, $\|f_n\|_p \leq M\ \forall n \in \mathbb{N}$ and $\forall\ E \subseteq [0,1]$ measurable $\int_E f_n dx \to \int_E f dx$. Show that

$$f_n \xrightarrow{w} f \text{ in } L^p[0,1].$$

Proof: Let $h \in L^{p'}[0,1]$ $\left(\frac{1}{p}+\frac{1}{p'}=1\right)$. We can find $s_0 \in L^{p'}[0,1]$ simple such that

$$\|h-s_0\|_{p'} \leq \frac{\varepsilon}{4M}.$$

The hypothesis implies

$$\langle f_n, s\rangle \longrightarrow \langle f, s\rangle \quad \forall\, s \in L^{p'}[0,1] \text{ simple.}$$

Then we have

$$|\langle f_n-f, h\rangle| \leq \underbrace{|\langle f_n-f, h-s_0\rangle|}_{\leq \|f_n-f\|_p \|h-s_0\|_{p'} \leq 2M \frac{\varepsilon}{4M}} + \underbrace{|\langle f_n-f, s_0\rangle|}_{\leq \frac{\varepsilon}{2}\ n\geq n_0}$$

$$\Rightarrow |\langle f_n-f, h\rangle| \leq \varepsilon \quad \forall\, n\geq n_0$$

$$\Rightarrow \quad f_n \xrightarrow{w} f \text{ in } L^p[0,1].$$

<u>QED</u>

Exercise 13: If $\{u_n, u\}_{n\in\mathbb{N}} \subseteq L^1[0,1]$ and $u_n \xrightarrow{a.e} u$, is it true that $u_n \longrightarrow u$ in $L^1[0,1]$? Explain

Proof: <u>NO</u> $\quad u_n(t) = n\chi_{[0,\frac{1}{n}]}(t)$

Then $u_n \xrightarrow{a.e} 0$ but

$\|u_n\|_1 = 1 \ \forall\, n\geq 1$ hence $u_n \not\to 0$ in $L^1[0,1]$

<u>QED</u>

Exercise 14: Let X be a Banach space, $y^* \in X^*$, $\|y^*\|_* = 1$ and $x \in X$. Show that if $M = \ker y^*$, then

$$d(x, M) = |\langle y^*, x\rangle|$$

Proof: For every $u \in M$ we have

$$\|x-u\| \geq |\langle y^*, x-u\rangle| = |\langle y^*, x\rangle|,$$

$$\Rightarrow d(x, M) \geq |\langle y^*, x\rangle| .$$

Let $\varepsilon \in (0,1)$ and use the Riesz Lemma to find $u \in \overline{B}_1$ s.t

$$\langle y^*, u\rangle \geq 1-\varepsilon .$$

We set $y = x - \dfrac{\langle y^*, x\rangle}{\langle y^*, u\rangle} u$. We have

$$\langle y^*, y\rangle = \langle y^*, x\rangle - \langle y^*, x\rangle = 0,$$

$$\Rightarrow y \in M .$$

Therefore

$$d(x, M) \leq \frac{|\langle y^*, x\rangle|}{|\langle y^*, u\rangle|} \leq \frac{|\langle y^*, x\rangle|}{1-\varepsilon}$$

Let $\varepsilon \downarrow 0$ to obtain

$$d(x, M) \leq |\langle y^*, x\rangle|,$$

$$\Rightarrow d(x, M) = |\langle y^*, x\rangle| .$$

QED

Exercise 15: Let X be a Banach space, $C \subseteq X$ w-compact. Show that $\overline{\text{conv}}\, C$ is w-compact

Proof: Let $x^* \in X^*$. We have

$$\sup_{c \in C} \langle x^*, c \rangle = \sup_{c \in \overline{\text{conv}}\, C} \langle x^*, c \rangle$$
$$\|$$
$$\langle x^*, \hat{c} \rangle$$

So, by James thm $\overline{\text{conv}}\, C$ is w-compact

QED

Exercise 16: Consider $\{f_n, f\}_{n \in \mathbb{N}} \subseteq L^1[0,1]$ such that

$$f_n \xrightarrow{a.e} f \quad , \quad \|f_n\|_1 \longrightarrow \|f\|_1$$

Show that $f_n \longrightarrow f$ in $L^1[0,1]$.

Proof: We have

$$|f_n| + |f| - |f_n - f| \xrightarrow{a.e} 2|f|$$

By Fatou's lemma we have

$$2\int_0^1 |f|\, dx \leq \liminf_{n \to \infty} \int_0^1 \left[|f_n| + |f| - |f_n - f| \right] dx$$

$$= 2\int_0^1 |f|\, dx - \limsup_{n \to \infty} \int_0^1 |f_n - f|\, dx,$$

$$\Rightarrow \limsup \int_0^1 |f_n - f|\, dx \leq 0,$$

$$\Rightarrow f_n \longrightarrow f \text{ in } L^1[0,1].$$

QED

Exercise 17: Suppose $Y \subseteq X$ subspace and assume that $\exists\, P \in \mathcal{L}(X,Y)$ projection. Show that Y is closed

Proof: Let $\{y_n\}_{n\in\mathbb{N}} \subseteq Y$ and $y_n \longrightarrow y$.

$$\text{Then } \underset{\substack{\| \\ y_n}}{P(y_n)} \longrightarrow P(y) \Rightarrow P(y) = y \Rightarrow y \in Y \Rightarrow Y \text{ is closed}$$

QED

Exercise 18: Let X, Y be Banach spaces, $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}(X,Y)$ is a sequence such that $A_n(x) \longrightarrow A(x)$ in Y $\forall x \in X$. Show that

"$x_n \longrightarrow x$ in $X \Rightarrow A_n(x_n) \longrightarrow A(x)$ in Y"

Proof: From the Banach-Steinhaus theorem, we have $A \in \mathcal{L}(X,Y)$ and $\exists\, M > 0$ s.t $\|A_n\|_{\mathcal{L}} \leq M$ $\forall n \in \mathbb{N}$. We have

$$\begin{aligned}
\|A_n(x_n) - A(x)\|_Y &\leq \|A_n(x_n - x)\|_Y + \|A_n(x) - A(x)\|_Y \\
&\leq \|A_n\|_{\mathcal{L}} \|x_n - x\|_X + \|A_n(x) - A(x)\|_Y \\
&\leq M \|x_n - x\| + \|A_n(x) - A(x)\|_Y ,
\end{aligned}$$

$$\Rightarrow A_n(x_n) \longrightarrow A(x) \text{ in } Y.$$

QED

Exercise 19: Let X be an infinite dimensional Banach space and $A \in \mathcal{L}_c(X)$. Show that $\exists\, h \in X$ such that

$$A(x) = h$$

has no solution $x \in X$

Proof: Arguing by contradiction suppose that $\forall h \in X$ the equation $A(x)=h$ has a solution. Hence $A(\cdot)$ is surjective. So, by the "open mapping theorem" we can find $\varepsilon > 0$ s.t

$$\varepsilon B_1 \subseteq A(B_1) \subseteq \overline{A(B_1)}$$

and $\overline{A(B_1)} \subseteq X$ compact. Hence

$$\overline{B}_1 \subseteq X \text{ compact},$$

$$\Rightarrow \dim X < \infty \Rightarrow \Leftarrow.$$

QED

Exercise 20: Let X be a reflexive subspace of ℓ^1. Show that $\dim X < \infty$.

Proof: Since X is reflexive, we have

$$\overline{B}_1^X = \text{w-compact},$$

$$\Rightarrow \overline{B}_1^X = \text{compact} \quad (\text{Schur property}),$$

$$\Rightarrow \dim X < \infty.$$

QED

Theorem: If X is compact, Y is Hausdorff, $f: X \to Y$ continuous, bijection, then $f(\cdot)$ is a homeomorphism

ℓ^p, ℓ^∞ $(1 \leq p < \infty)$.

$c_0 \subseteq \ell^\infty$ $(u_n \to 0)$, $c \subseteq \ell^\infty$, $c_{00} \subseteq \ell^\infty$ (finite suppor

$$c_0 = \overline{c_{00}}^{\|\cdot\|_\infty}$$

Exercise 21: Let X be an incomplete normed space and $C \subseteq X$ compact. Show that $\overline{\text{conv}}\, C$ need not be compact.

Proof: Let $C = \{\frac{1}{n} e_n\} \cup \{0\} \subseteq c_{00}$. This set is compact. If $\overline{\text{conv}}\, C$ in c_{00} were compact, it would be equal to $\overline{\text{conv}} C$ in c_0. Note that $(\frac{1}{2^n} \frac{1}{n}) \in \overline{\text{conv}}\, C$ in c_0 but not in c_{00}.

QED

If X = normed space and $A, C \subseteq X$ convex sets,

then $\text{conv}(A \cup C) = \{ ta + (1-t)c : a \in A,\ c \in C,\ 0 \leq t \leq 1 \}$

Exercise 22: Let X, Y be Banach spaces and $A \in \mathcal{L}(X,Y)$. Suppose

$$\|A(x)\|_Y \geq c \|x\|_X \quad \text{for some } c > 0, \text{ all } x \in X.$$

Show that $R(A)$ is closed.

Proof: Let $\{y_n\}_{n \in \mathbb{N}} \subseteq R(A)$ and assume that $y_n \to y$. Then $y_n = A(x_n)$ and for $n, m \in \mathbb{N}$ we have

$$\|y_n - y_m\|_Y = \|A(x_n - x_m)\|_Y \geq c \|x_n - x_m\|_X,$$

$\Rightarrow \{x_n\}_{n \in \mathbb{N}} \subseteq X$ is Cauchy,

$\Rightarrow x_n \to x$

$$\Rightarrow A(x_n) = y_n \longrightarrow y = A(x),$$

$\Rightarrow y \in R(A)$ and so $R(A)$ is closed.

QED

<u>Exercise 23:</u> Let X, Y be Banach spaces and $A \in \mathcal{L}_c(X,Y)$. Suppose that $u_n \xrightarrow{w} u$ in X. Show that $A(u_n) \to A(u)$ in Y.

<u>Proof:</u> Recall that $A \in \mathcal{L}(X_w, Y_w)$. Therefore

$$A(u_n) \xrightarrow{w} A(u) \text{ in } Y.$$

Also $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ is bounded. Since $A \in \mathcal{L}_c(X,Y)$ we have

$$\{A(u_n)\}_{n\in\mathbb{N}} \subseteq Y \text{ is relatively compact}$$

So, we can find a subsequence s.t

$$A(u_{n_k}) \longrightarrow y \text{ in } Y.$$

$$\Rightarrow y = A(u).$$

Then by the Urysohn criterion we have

$$A(u_n) \longrightarrow A(u) \text{ in } Y.$$

QED

<u>Remark:</u> The converse is <u>not</u> in general true. Consider $id: \ell^1 \to \ell^1$

<u>Exercise 24:</u> Let X, Y be Banach spaces. and $A \in \mathcal{L}(X,Y)$

Let $|u| = \left[\|u\|_X^2 + \|A(u)\|_Y^2\right]^{1/2} \quad \forall u \in X.$

Show that $\|\cdot\|_X$, $|\cdot|$ are equivalent norms

Proof: Need to show that $\exists\ 0 < c_1 < c_2$ s.t

$$c_1 \|u\|_X \leq |u| \leq c_2 \|u\|_X \quad \forall u \in X.$$

Evidently we have

$$\|u\|_X \leq |u| \quad \forall u \in X.$$

Suppose that we can find $u_n \in X$ s.t

$$n \|u_n\|_X < |u_n|,$$

$$\Rightarrow (n-1) \|u_n\|_X < \|A(u_n)\|_Y,$$

$$\Rightarrow (n-1) < \|A(v_n)\|_Y \qquad v_n = \frac{u_n}{\|u_n\|_X} \quad (\Rightarrow \|v_n\|_X = 1)$$

Since $\{A(v_n)\}_{n \in \mathbb{N}} \subseteq Y$ is bounded we have a contradiction.

QED

Exercise 25: Let X, Y be Banach spaces and $A \in \mathcal{L}(X,Y)$. Show that $R(A) \subseteq Y$ is dense iff A^* is 1-1

Proof: $\Rightarrow$ Arguing by contradiction, suppose that A^* is not 1-1. Then we can find $y^* \in Y^*$, $y^* \neq 0$ s.t $A^*(y^*) = 0$. Then

$$\langle A^*(y^*), x \rangle = 0 \quad \forall x \in X,$$

$$\Rightarrow \langle y^*, A(x) \rangle = 0 \quad \forall x \in X,$$

$$\Rightarrow \quad y^* \in R(A)^\perp = \{0\} \text{ (since } R(A) \text{ dense in } Y) \Rightarrow\Leftarrow$$

$\Leftarrow$ Again by contradiction. Suppose $\overline{R(A)} \neq Y$. Then we can find $y^* \in Y^*$, $y^* \neq 0$ s.t

$$y^*\Big|_{R(A)} = 0 \;,$$

$$\Rightarrow \langle y^*, A(x) \rangle = 0 \quad \forall x \in X,$$

$$\Rightarrow \langle A^*(y^*), x \rangle = 0 \quad \forall x \in X,$$

$$\Rightarrow A^*(y^*) = 0 \quad \text{and} \quad y^* \neq 0,$$

$$\Rightarrow A^* \text{ is not } 1\text{-}1, \Rightarrow\Leftarrow$$

QED

Exercise 26: Let X be a normed space, $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ Cauchy and $x_n \xrightarrow{w} 0$ in X. Show that $x_n \to 0$.

Proof: Since $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is Cauchy given $\varepsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that

$$x_n - x_m \in \varepsilon \overline{B}_1 \quad \forall n, m \geq n_0$$

$$\Rightarrow x_n \in x_m + \varepsilon \overline{B}_1$$

Fix $m \geq n_0$. The set $x_m + \varepsilon\overline{B}_1$ is closed, convex. Thus w-closed (Mazur's Lemma). Hence

$$x_m \in \varepsilon \overline{B}_1 \quad \forall m \geq n_0$$

$$\Rightarrow x_n \longrightarrow 0 \text{ in } X .$$

QED

Exercise 27: Let X be a Banach space and $C \subseteq X^*$ w^*-compact. Is $\overline{\text{conv}}^{w^*} C$ w^*-compact? Explain

Proof: YES Note that C is bounded. So, we can find $r>0$ big such that

$$C \subseteq \overline{B}_r,$$

$$\Rightarrow \overline{\text{conv}}^{w^*} C \subseteq \overline{B}_r$$

($\overline{B}_r$ is w^*-closed, convex. Let $\{x_\alpha^*\}_{\alpha \in J} \subseteq \overline{B}_r$ $x_\alpha^* \xrightarrow{w^*} x^*$. Then $\langle x_\alpha^*, x\rangle \to \langle x^*, x\rangle$ $\forall x \in X$. Then

$$|\langle x_\alpha^*, x\rangle| \leq \|x_\alpha^*\|_* \|x\| \leq r\|x\|,$$

$$\Rightarrow |\langle x^*, x\rangle| \leq r\|x\| \quad \forall x \in X$$

$$\Rightarrow \|x^*\|_* \leq r, \text{ that is, } x^* \in \overline{B}_r.)$$

But $\overline{B}_r$ is w^*-compact (Alaoglu). So

$$\overline{\text{conv}}^{w^*} C \subseteq X^* \text{ is } w^*\text{-compact}$$

QED

Suppose P is a property defined on normed spaces.

X = normed space, $Y \subseteq X$ closed subspace

" $X, Y, X/Y$ two of these have P

$\Downarrow$

the third has P "

Then we say "P is a three space property"

Ex: Completeness, Separability, Reflexivity

Exercise 28: Let X be a reflexive Banach space, Y a Banach space and $A \in \mathcal{L}(X,Y)$ is surjective. Show that Y is reflexive.

Proof: Consider $\hat{A} : X/_{N(A)} \longrightarrow Y$ defined by

$$\hat{A}([x]) = A(x) \quad \forall x \in X.$$

This is an isomorphism and $X/_{N(A)}$ = reflexive (three spaces property). So Y = reflexive

QED