EXERCISES 3

Exercise 1: Show that $L^1(\mathbb{R})$ is not reflexive

Proof: We will show that the canonical embedding

$$j: L^1(\mathbb{R}) \longrightarrow L^\infty(\Omega)^*$$

is not surjective.

Consider the Dirac functional defined by

$$\delta(g) = g(0) \quad \forall g \in C_b(\mathbb{R})$$

This is a continuous linear functional on $C_b(\mathbb{R})$ of norm 1

By the Hahn-Banach theorem it extends to a continuous linear functional on $L^\infty(\mathbb{R})$

Claim: There is no function $f \in L^1(\mathbb{R})$ such that

$$\int_{\mathbb{R}} fg\,dx = g(0) \quad \forall g \in C_b(\mathbb{R}) \qquad (*)$$

Arguing by contradiction suppose that such a function f exists. Let $g_n(x) = \min\{n|x|, 1\}$ $x \in \mathbb{R}$ Then $\{g_n\}_{n\in\mathbb{N}} \subseteq C_b(\mathbb{R})$ and

$$g_n(0) = 0 \ , \ fg_n \xrightarrow{a.e} f \ , \ |fg_n| \leq f$$

So, by the Lebesgue Dominated Convergence Theorem we have

$$\int_{\mathbb{R}} f\,dx = \lim_{n\to\infty} \int_{\mathbb{R}} fg_n\,dx = \lim_{n\to\infty} g_n(0) = 0 .$$

But in $(*)$ let $g \equiv 1$. Then $\int_{\mathbb{R}} f\,dx = 1$, a contradiction.

QED

Exercise 2: Let $1<p<\infty$, $\{f_n\}_{n\in\mathbb{N}} \subseteq L^p(\Omega)$ is bounded and

$$f_n \xrightarrow{a.e} f.$$

Show that $f_n \xrightarrow{w} f$ in $L^p(\Omega)$

Proof: First we show that $f \in L^p(\Omega)$. Indeed by Fatou's lemma we have

$$\int_\Omega |f|^p dx \leq \liminf_{n\to\infty} \int_\Omega |f_n|^p dx \leq M.$$

$$\Rightarrow f \in L^p(\Omega).$$

We may assume $f=0$. Just replace f_n by $f_n - f$, $n\in\mathbb{N}$.

For $m\in\mathbb{N}$ let

$$C_m = \{x\in\Omega : |f_n(x)| \leq 1 \quad \forall\, n\geq m\},$$

$$K_m = \{g \in L^{p'}(\Omega) : g\big|_{\Omega\setminus C_m} = 0\}.$$

Since $f_n \xrightarrow{a.e} 0$, almost all $x\in\Omega$ belongs in $\bigcup_{m\in\mathbb{N}} C_m$

Notice that $\{C_m\}_{m\in\mathbb{N}}$ is nondecreasing. Then $\bigcup_{m\geq 1} K_m \subseteq L^{p'}(\Omega)$ is dense.

To see this let $g \in L^{p'}(\Omega)$. Then $g_m = \chi_{K_m} g \longrightarrow g$ in $L^{p'}(\Omega)$, $g_m \in K_m$ $m\in\mathbb{N}$.

So to show that $f_n \xrightarrow{w} 0$ in $L^p(\Omega)$ it suffices to show

$$\int_\Omega f_n g\, dx \longrightarrow 0 \qquad \forall\, g \in \bigcup_{m\geq 1} K_m \tag{**}$$

But $|f_n g| \leq g \in L^{p'}(\Omega) \subseteq L^1(\Omega)$ and $f_n g \xrightarrow{a.e} 0$. So using the dominated convergence theorem, we see that (**) holds. Hence

$$f_n \xrightarrow{W} f \text{ in } L^p(\Omega).$$

QED

Exercise 3: Let H be a Hilbert space and $A \in \mathcal{L}(H)$. Show that $H = N(A) \oplus R(A^*)$.

Proof: It suffices to show that $R(A^*)^\perp = N(A)$.

We have

$$N(A) = \{x \in H : A(x) = 0\} = \{x \in H : (A(x), h) = 0 \ \forall h \in H\}$$

$$= \{x \in H : (x, A^*(h)) = 0 \ \forall h \in H\}$$

$$= R(A^*)^\perp.$$

QED

Exercise 4: Let H be a Hilbert space, $A \in \mathcal{L}(H)$, $\|A\|_{\mathcal{L}} \leq 1$. Show that $I - A^*A \geq 0$.

Proof: We have

$$(I - A^*A)^* = I - (A^*A)^* = I - A^*A,$$

$$\Rightarrow I - A^*A \text{ is self-adjoint}$$

For any $x \in H$, we have

$$((I - A^*A)(x), x) = \|x\|^2 - (A^*A(x), x)$$

$$= \|x\|^2 - \|A(x)\|^2$$

$$\geq \|x\|^2 - \|A\|_{\mathcal{L}}^2 \|x\|^2 \geq 0.$$

QED

Exercise 5: Let X, Y be Banach space and $A : X \to Y$ additive (that is, $A(x+u) = A(x) + A(u)$ $\forall x, u \in X$) and continuous at 0. Show that $A \in \mathcal{L}(X, Y)$.

<u>**Proof:**</u> On account of the additivity property we have

$$A(0) = 2A(0)$$

$$\Rightarrow A(0) = 0$$

Let $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ and assume that $x_n \to x$. Then $x_n - x \to 0$

Exploiting the continuity of $A(\cdot)$ at 0, we have

$$A(x_n - x) \to A(0) = 0,$$

$$\Rightarrow A(x_n) = A(x_n - x) + A(x) \to A(x),$$

$$\Rightarrow A(x_n) \to A(x).$$

Let $x \in X$, $\lambda \in \mathbb{R}$. We have

$$0 = A(0) = A(x-x) = A(x) + A(-x),$$

$$\Rightarrow A(-x) = -A(x).$$

Using the additivity of $A(\cdot)$, we have

$$A(mx) = mA(x) \quad \forall m \in \mathbb{Z}.$$

Let $n \in \mathbb{N}$, $m \in \mathbb{Z}$. Then

$$mA(x) = A(mx) = A\left(n \frac{m}{n} x\right) = nA\left(\frac{m}{n} x\right),$$

$$\Rightarrow A\left(\frac{m}{n} x\right) = \frac{m}{n} A(x),$$

$$\Rightarrow A(qx) = qA(x) \quad \forall q \in \mathbb{Q}$$

Consider $\{q_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ s.t $q_n \to \lambda$. Then

$$q_n x \to \lambda x$$

$$\Rightarrow A(q_n x) \to A(\lambda x)$$

$\Rightarrow q_n A(x) \longrightarrow A(\lambda x)$,

$\Rightarrow \lambda A(x) = A(\lambda x) \quad \forall x \in X \quad \forall \lambda \in \mathbb{R}$,

$\Rightarrow A \in \mathcal{L}(X, Y)$.

QED

Proposition: If X is a Banach space and $A \in \mathcal{L}(X)$ with

$$\|A\|_{\mathcal{L}} < 1.$$

then $I - A$ is an isomorphism, $(I-A)^{-1} = \sum_{n \geq 0} A^n$

Exercise 6: Let X be a Banach space and $\{A_n\}_{n \geq 1} \subseteq \mathcal{L}(X)$ are isomorphisms such that $\|A_n^{-1}\|_{\mathcal{L}} < 1 \quad \forall n \in \mathbb{N}$ and $A_n \xrightarrow{\|\cdot\|_{\mathcal{L}}} A$. Show that A is an isomorphism

Proof: We can find $n_0 \in \mathbb{N}$ such that

$$\|A_n - A\|_{\mathcal{L}} < 1 \quad \forall n \geq n_0$$

Then for all $n \geq n_0$, we have

$$\|I - A_n^{-1} A\|_{\mathcal{L}} = \|A_n^{-1}(A_n - A)\|_{\mathcal{L}} \leq \|A_n^{-1}\|_{\mathcal{L}} \|A_n - A\|_{\mathcal{L}} < 1,$$

$\Rightarrow A_{n_0}^{-1} A$ isomorphism,

$\Rightarrow A = A_{n_0} A_{n_0}^{-1} A$. is an isomorphism too

QED

Exercise 7: Let X, Y be Banach spaces and $A \in \mathcal{L}(X,Y)$ is surjective. Consider $C \subseteq X$ nonempty. Show that

$A(C) \subseteq Y$ is closed iff $N(A) + C \subseteq X$ is closed

Proof: $\Rightarrow$ We have

$$N(A) + C = A^{-1}(A(C))$$

But $A^{-1}(A(C))$ is closed since $A \in \mathcal{L}(X,Y)$. Therefore

$$N(A) + C \text{ is closed}$$

$\Leftarrow$ Since by hypothesis $A(\cdot)$ is surjective

$$A((N(A)+C)^c) = A(C)^c$$

But $(NA+C)^c \subseteq X$ open and since A is surjective by the Open Mapping Theorem we have that

$$A(C)^c \subseteq Y \text{ is open},$$

$$\Rightarrow A(C) \subseteq Y \text{ is closed.}$$

QED

Exercise 8: Let H be a Hilbert space and $A \in \mathcal{L}(H)$ is normal. Show that "$A^2 \in \mathcal{L}_c(H) \Rightarrow A \in \mathcal{L}_c(H)$"

Proof: First we show that

$$\|A(u)\|^2 \leq \|A^2(u)\| . \|u\| \quad \forall u \in H$$

Since A is normal, we have

$$(AA^*(u), u) = (A^*A(u), u) \quad \forall u \in H$$

$$\Rightarrow \|A(u)\| = \|A^*(u)\| \quad \forall u \in H. \qquad (*)$$

We have

$$\|A(x)\|^2 = (A(x), A(x)) = (A^*A(x), x)$$

$$\leq \|A^*A(x)\| \, \|x\|$$

$$\leq \|A^2(x)\| \, \|x\|$$

(use $(*)$ with $u = A(x)$)

Now let $\{x_n\}_{n\in\mathbb{N}} \subseteq \overline{B}_1$. We have

$$\|A(x_n) - A(x_m)\|^2 \leq \|A^2(x_n) - A^2(x_m)\| \, \|x_n - x_m\|$$

$$\leq 2\|A^2(x_n) - A^2(x_m)\| \quad \forall n, m \geq 1$$

Therefore if $\{A^2(x_n)\}_{n\in\mathbb{N}}$ is Cauchy, so is $\{A(x_n)\}_{n\in\mathbb{N}}$

By hypothesis $A^2 \in \mathcal{L}_c(H)$. So, we may assume that

$$\{A^2(x_n)\}_{n\in\mathbb{N}} \subseteq H \text{ is convergent,}$$

$$\Rightarrow \{A(x_n)\}_{n\in\mathbb{N}} \subseteq H \text{ convergent,}$$

$$\Rightarrow A \in \mathcal{L}_c(H).$$

QED

Proposition: If X is a Hausdorff topological space then the F.A.E

(a) X is compact.

(b) Every family of nonempty closed sets with the the finite intersection property has a nonempty intersection.

(c) Every net has a convergent subnet.

Exercise 9: Let X be a Banach space.

Show that the following two properties are equivalent

(a) X is reflexive.

(b) For every decreasing sequence $\{C_n\}_{n\in\mathbb{N}}$ of nonempty closed, convex and bounded sets, we have

$$\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$$

Proof: (a) $\Rightarrow$ (b) So, let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence as in (b)

For every $n\in\mathbb{N}$ C_n is w-compact. Then

$$C_n \subseteq C_1 \quad \forall n\in\mathbb{N}.$$

$\{C_n\}_{n\in\mathbb{N}}$ has the finite intersection property

$$\Downarrow$$

$$\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$$

(b) $\Rightarrow$ (c) Let $x^* \in X^* \setminus \{0\}$ and set

$$C_n = \{ x\in X : \|x\| \leq 1 \ , \ \langle x^*, x\rangle \geq \|x^*\|_{*} - \tfrac{1}{n} \}$$

Then the family $\{C_n\}_{n\geq 1}$ is decreasing and consists of nonempty, closed, convex, bounded sets. So, by hypothesis

$$\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$$

Let $x \in \bigcap_{n\in\mathbb{N}} C_n$. Then

$\|x\| \leq 1$ and $\langle x^*, x\rangle \geq \|x^*\|_* - \frac{1}{n}$ $\forall n \in \mathbb{N}$

$\Rightarrow \|x\| \leq 1$, $\langle x^*, x\rangle = \|x^*\|_*$

Since $x^* \in X^* \setminus \{0\}$ arbitrary by James' thm X = Reflexive

QED

Exercise 10: Let X, Y be Banach spaces and $A \in \mathcal{L}(X,Y)$ is a bijection. Show that $A^* \in \mathcal{L}(Y^*, X^*)$ is a bijection too and $(A^*)^{-1} = (A^{-1})^*$

Proof: From Banach's Theorem (Theorem 3.2.10) we have

$$A^{-1} \in \mathcal{L}(Y, X)$$

We have

$$AA^{-1} = I_Y \quad \text{and} \quad A^{-1}A = I_X$$

$$\Rightarrow (A^{-1})^* A^* = I_{Y^*} \quad \text{and} \quad A^*(A^{-1})^* = I_{X^*}$$

$\Rightarrow A^*$ is isomorphism, $(A^*)^{-1} = (A^{-1})^*$

QED

Exercise 11: Let H be a Hilbert space, $A \in \mathcal{L}(H)$ and

$$\langle A(u), u \rangle \geq c\|u\|^2 \quad \text{for some } c>0, \text{ all } u \in H.$$

Show that A is an isomorphism

Proof: From the hypothesis on $A(\cdot)$ we have

$$R(A) \subseteq H \text{ closed}, \quad N(A) = \{0\}.$$

Also we have

$$\langle u, A^*(u) \rangle \geq c\|u\|^2,$$

$$\Rightarrow \|A^*(u)\| \geq c\|u\| \quad \forall u \in H,$$

$$\Rightarrow N(A^*) = \{0\}$$

We know that $H = N(A^*)^{\perp} = \overline{R(A)}$. Hence

$$R(A) = H,$$

$$\Rightarrow A \text{ is a bijection},$$

$$\Rightarrow A^{-1} \in \mathcal{L}(H) \quad \text{(Banach Thm)}.$$

QED

Exercise 12: Show that Mazur's lemma is not true for the w^*-topology.

Proof: Consider X a nonreflexive Banach space. Then the set $\overline{B}_1^X = \{u \in X : \|u\| \leq 1\}$ is convex, closed in X^*. However, it is not w^*-closed since $\overline{\left(\overline{B}_1^X\right)}^{w^*} = \overline{B}_1^{X^{**}}$ (Goldstine Thm).

QED

<u>**Exercise 13:**</u> Let X be a Banach space, $Y \subseteq X$ a finite dimensional subspace and $u \in X$. Show that $\exists\, \hat{y} \in Y$ s.t $d(u,Y) = \inf\left[\|u-y\| : y \in Y\right] = \|u-\hat{y}\|$

<u>Proof:</u> We know that Y is closed. Let $\{y_n\}_{n\in\mathbb{N}} \subseteq Y$ be a minimizing sequence, that is,

$$\|u-y_n\| \downarrow d(u,Y).$$

Then $\{y_n\}_{n\in\mathbb{N}} \subseteq Y$ bounded and because Y is finite dimensional, we may assume that

$$y_n \longrightarrow \hat{y} \in Y$$

Then $\|u-y_n\| \longrightarrow \|u-\hat{y}\| = d(u,Y)$.

QED

<u>**Exercise 14:**</u> Let $E \subseteq L^1(\Omega)$ be relatively w-compact and consider

$$C = \{u \in L^1(\Omega) : |u(z)| \leq |y(z)| \text{ for a.a } z \in \Omega \text{ some } y \in E\}$$

Show that C is relatively w-compact.

<u>Proof:</u> We know that E is bounded, hence C is bounded too.

By hypothesis E is uniformly integrable. So, given $\varepsilon > 0$, we can find $\delta > 0$ s.t

$$|A|_N \leq \delta \Rightarrow \int_A |y|\,dx \leq \varepsilon \quad \forall y \in E$$

$\Rightarrow |A|_N \leq \delta \Rightarrow \int_A |u| dx \leq \varepsilon \quad \forall u \in C,$

$\Rightarrow$ C = uniformly integrable,

$\Rightarrow$ $C \subseteq L^1(\Omega)$ is relatively w-compact

(Dunford-Pettis Thm)

QED

Exercise 15: $\{f_n, g_n, h_n\}_{n\in\mathbb{N}} \subseteq L^1(\Omega)$, $f_n \leq h_n \leq g_n$ a.e

$f_n \xrightarrow{a.e} f$, $h_n \xrightarrow{a.e} h$, $g_n \xrightarrow{a.e} g$. Suppose that $f, g \in L^1(\Omega)$

and $\int_\Omega f_n dx \to \int_\Omega f dx$, $\int_\Omega g_n dx \to \int_\Omega g dx$.

Show that $h \in L^1(\Omega)$ and $\int_\Omega h_n dx \to \int_\Omega h dx$

Proof: Clearly we have

$$f(x) \leq h(x) \leq g(x) \quad \text{a.e.},$$

$$\Rightarrow h \in L^1(\Omega).$$

Moreover, we have

$$|h(z)| \leq |f(z)| + |g(z)| \quad \text{for a.a } z \in \Omega$$

So, by Fatou's lemma we have

$$\int_\Omega h dx - \int_\Omega f dx = \int_\Omega (h-f) dx$$

$$= \int_\Omega \lim (h_n - f_n) dx$$

$$\leq \liminf \int_\Omega (h_n - f_n) dx$$

$$= \liminf \int_\Omega h_n dx - \int_\Omega f dx,$$

$$\Rightarrow \int_\Omega h\, dx \le \liminf \int_\Omega h_n dx .$$

In a similar fashion, using this time the sequence $\{g_n\}_{n\in\mathbb{N}}$ we show that

$$\limsup_{n\to\infty} \int_\Omega h_n dx \le \int_\Omega h\, dx .$$

Therefore finally we have

$$\int_\Omega h_n dx \longrightarrow \int_\Omega h\, dx .$$

QED

Exercise 16: Suppose $\{f_n\}_{n\in\mathbb{N}} \subseteq L^1(\Omega)$, $f_n \geq 0$, $f_n \xrightarrow{a.e} f$ with $f \in L^1(\Omega)$ and $\int_\Omega f_n dx \to \int_\Omega f dx$. Show that $\|f_n - f\|_1 \to 0$

Proof: We have

$$0 \le |f_n - f| \le |f_n| + |f| = f_n + f$$

Use Exercise 15.

QED

Exercise 17: If H is a Hilbert space and $A \in \mathcal{L}(H)$, show that

$$\|A^* A\|_{\mathcal{L}} = \|A\|_{\mathcal{L}}^2$$

Proof: We have

$$\|A^*A\|_{\mathcal{L}} \leq \|A^*\|_{\mathcal{L}} \|A\|_{\mathcal{L}} = \|A\|_{\mathcal{L}}^2$$

On the other hand

$$\|A(x)\|^2 = (A(x), A(x)) = (A^*A(x), x)$$
$$\leq \|A^*A(x)\| \|x\|$$
$$\leq \|A^*A\|_{\mathcal{L}} \|x\|^2. \quad \forall x \in H,$$

$$\Rightarrow \|A\|_{\mathcal{L}}^2 \leq \|A^*A\|_{\mathcal{L}},$$

$$\Rightarrow \|A^*A\|_{\mathcal{L}} = \|A\|_{\mathcal{L}}^2$$

QED

Exercise 18: Let H be a Hilbert space. Show that the set of self-adjoint operators is closed in $\mathcal{L}(H)$.

Proof. Consider a sequence $\{A_n\}_{n\in\mathbb{N}}$ of s.a. operators and assum that $A_n \longrightarrow A$ in $\mathcal{L}(H)$. We have

$$\|A - A^*\|_{\mathcal{L}} \leq \|A - A_n\|_{\mathcal{L}} + \|A_n - A_n^*\|_{\mathcal{L}} + \|A_n^* - A^*\|_{\mathcal{L}}$$
$$= \|A - A_n\|_{\mathcal{L}} + \|(A_n - A)^*\|_{\mathcal{L}} \quad (\text{sine } A_n \text{ s.a})$$
$$= 2\|A - A_n\|_{\mathcal{L}} \longrightarrow 0$$

$$\Rightarrow A = A^*.$$

QED

Exercise 19: Let X be a Banach space and $A \in \mathcal{L}_c(X)$ with $I-A$ injective. Show that $I-A$ has a continuous inverse on $(I-A)(X)$.

Proof: Arguing by contradiction, suppose that the assertion is not true. Then we can find $u_n \in X$ such that

$$\|(I-A)(u_n)\| < \frac{1}{n}\|u_n\|$$

Consider $\left\{ v_n = \frac{u_n}{\|u_n\|} \right\}_{n\in\mathbb{N}}$ Then

$$\|v_n\| = 1 \quad \forall n \in \mathbb{N}$$

So, $\exists$ a subsequence $\{v_{n_k}\}_{k\in\mathbb{N}}$ s.t

$$A(v_{n_k}) \longrightarrow y \text{ in } X$$

But $\|(I-A)(v_{n_k})\| < \frac{1}{n_k} \quad \forall k \in \mathbb{N}$,

$$\Rightarrow v_{n_k} \longrightarrow y \text{ in } X$$

$$\Rightarrow y = A(y), \quad \|y\| = 1$$

This contradicts that $A(\cdot)$ is 1-1.

QED

Exercise 20: Let X, Y be Banach spaces and $A : X \to Y$ linear operator which maps bounded sets of X to relatively compact subsets of Y. Show that $A \in \mathcal{L}_c(X,Y)$

Proof: We need to show that $A \in \mathcal{L}(X,Y)$.

Suppose that this is not true. Then we can find $u_n \in X$

s.t

$$\|A(u_n)\|_Y > n\|u_n\|_X$$

$$\Rightarrow \|A(v_n)\| > n \qquad v_n = \frac{u_n}{\|u_n\|}$$

But $\|v_n\| = 1$ so $\{A(v_n)\}_{n\in\mathbb{N}}$ relatively compact

thus bounded, contradiction.

QED

Exercise 21: Suppose $A \in \mathcal{L}(\ell^1, \ell^2)$ is injective. Is $A(\ell^1) \subseteq \ell^2$ closed? Explain.

Proof: If $V = A(\ell^1)$ is closed, the V is reflexive Hence by the Banach Theorem ℓ^1 and V are isomorphic $\Rightarrow$ ℓ^1 is reflexive, a contradiction

QED

Exercise 22: Let X be a Banach space and $C \subseteq X^*$ is w^* compact Consider $\overline{\text{conv}}^{w^*} C = \hat{C}$. Is $\hat{C}$ is w^*-compact. Explain.

Proof: YES. Note that C is bounded. Hence $\overline{\text{conv}}^{w^*} C = \hat{C}$ is bounded, w^*-closed. By the theorem of Alaoglu, it is w^*-compact

QED

Exercise 23: Let X, Y be Banach spaces and $A \in \mathcal{L}(X,Y)$. Show that $A^* \in \mathcal{L}(Y^*_{w^*}, X^*_{w^*})$

Proof: From the definition of the w^*-topology, it is sufficient to show that for every $u \in X$ $u \circ A^*: Y^* \to \mathbb{R}$ is w^*-continuous. But this is equivalent to saying that $u \circ A^* \in Y^*$ which is true.

QED

Exercise 24: Let $\Omega \subseteq \mathbb{R}^N$ be bounded. Show that $L^1(\Omega)$ is not reflexive.

Proof: Arguing by contradiction, suppose that $L^1(\Omega)$ is reflexive. Since $L^1(\Omega)$ is also separable, then $L^1(\Omega)^{**}$ is separable too. So, $L^1(\Omega)^*$ is separable. But $L^1(\Omega)^* = L^\infty(\Omega)$ and the latter is nonseparable. So $L^1(\Omega)$ is not reflexive.

QED