SOLUTIONS PS 3

PROBLEM 1: By hypothesis id-A is injective.

Suppose that id-A does not have a continuou inverse. Then we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that

$$\|(id-A)(u_n)\| < \frac{1}{n}\|u_n\|$$

$$\Rightarrow \left\|(id-A)\left(\frac{u_n}{\|u_n\|}\right)\right\| < \frac{1}{n}$$

Note if $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\|=1$ and so since $A\in\mathcal{L}_c(X)$ we may assume $A(v_n)\longrightarrow y$ in X. Then

$$v_n \longrightarrow y \text{ in } X$$

$$\Rightarrow A(y)=y \quad \|y\|=1.$$

a contradiction to the hypothesis id-A injective.

QED

REMARK: The statement of the problem is not correct

The injectivity of id-A <u>is</u> a hypothesis!

PROBLEM 2: Note

$$R(A) = \bigcup_{n\in\mathbb{N}} nA(\overline{B}_1)$$

Since $A(\overline{B}_1)$ is compact, it is separable. So

$R(A)$ is separable.

QED

PROBLEM 3: We can $n_0 \in \mathbb{N}$ such that

$$\|A_n - A\|_{\mathcal{L}} < 1 \quad \forall n \geq n_0$$

Then for $n \geq n_0$ we have

$$\| id - A_n^{-1} A\|_{\mathcal{L}} = \| A_n^{-1} (A_n - A)\|_{\mathcal{L}} \leq \|A_n^{-1}\|_{\mathcal{L}} < 1,$$

$\Rightarrow A_{n_0}^{-1} A$ is an isomorphism,

$\Rightarrow A_{n_0} (A_{n_0}^{-1} A) = A$ an isomorphism.

QED

PROBLEM 4: $\Leftarrow$ $(AB)^{-1} = B^{-1} A^{-1} \in \mathcal{L}(X) \Rightarrow AB$ invertible

$\Rightarrow$ If for $u \neq 0$ $B(u) = 0$, then

$$(AB)(u) = A(0) = 0,$$

$\Rightarrow AB$ is not injective, a contradiction.

So, B is injective.

Similarly for A using that $AB = BA$

If A is not surjective, then AB is not too a contradiction. Similarly if $B(X) \neq X$, using that $AB = BA$

So, A, B are surjective, hence by the Banach Theorem A, B are invertible.

QED

PROBLEM 5: Let $u_n \to u$ in X, $A(u_n) \to y^*$ in X^*. We have

$$\langle A(u_n) - A(v), u_n - v \rangle \geq 0 \quad v \in X,$$

$$\Rightarrow \langle y^* - A(v), u - v \rangle \geq 0 \quad \forall v \in X$$

Let $v = u + \lambda h$ $\lambda > 0$, $h \in X$. Then

$$\lambda \langle y^* - A(u), h \rangle \leq \lambda^2 \langle A(h), h \rangle,$$

$$\Rightarrow \langle y^* - A(u), h \rangle \leq \lambda \langle A(h), h \rangle,$$

$$\Rightarrow \langle y^* - A(u), h \rangle \leq 0 \quad \forall h \in X,$$

$$\Rightarrow y^* = A(u).$$

So by the closed graph theorem, we have

$$A \in \mathcal{L}(X, X^*).$$

QED