EXERCISES 2

Exercise 11: Let X be a Banach space, $A \subseteq X$ and assume that for every $Y \subseteq X$ separable, closed subspace of X we have that $A \cap Y$ is w-compact. Show that A is w-compact

Proof: Consider $\{u_n\}_{n\in\mathbb{N}} \subseteq A$. Let $Y := \overline{\operatorname{span}\{u_n\}_{n\in\mathbb{N}}}$. Evidently

$Y \subseteq X$ separable, closed subspace

So, by hypothesis $A \cap Y$ is w-compact. Since $\{u_n\}_{n\in\mathbb{N}} \subseteq A \cap Y$, by the Eberlein-Smulian theorem $\exists \{u_{n_k}\}_{k\in\mathbb{N}}$ subsequence of $\{u_n\}_{n\in\mathbb{N}}$ s.t $u_{n_k} \xrightarrow{w} u$ in Y and so in X too. Notice that $u \in A \cap Y$ and so by Eberlein-Smulian A is w-compact

QED

Exercise 12: Let X be a Banach space, $Y \subseteq X$ a closed subspace and $x_0 \notin Y$. Show that $\operatorname{span}\{Y, x_0\} \subseteq X$ is closed

Proof: Suppose $\{u_n\}_{n\in\mathbb{N}} \subseteq \operatorname{span}\{Y, x_0\}$ and assume $u_n \to u$ in X.

We have

$$u_n = y_n + \xi_n x_0 \quad \text{with } y_n \in Y,\ \xi_n \in \mathbb{R},\ n\in\mathbb{N}.$$

We claim that $|\xi_n| \leq \|u_n\| \quad \forall n \in \mathbb{N}$

Indeed by the strong separation theorem (see also Prop. 3.1.62) we can find $x_n^* \in X^*$ $\|x_n^*\|_* = 1$ and $x_n^*\big|_Y = 0$, $\langle x_n^*, x_0 \rangle = 1$ $\forall n \in \mathbb{N}$. Then

$$\|u_n\| \geq |\langle x_n^*, u_n \rangle| = |\xi_n| \, |\langle x_n^*, x_0 \rangle| = |\xi_n| \quad \forall n \in \mathbb{N}$$

This proves our Claim.

It follows that $\{\xi_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is bounded and so we may assume that $\xi_n \to \xi$ in $\mathbb{R}$. Then

$$u_n - \xi_n x_0 \longrightarrow u - \xi x_0,$$

$$\Rightarrow \quad y_n \longrightarrow u - \xi x_0 \in Y,$$

$$\Rightarrow \quad u = y + \xi x_0,$$

$$\Rightarrow \quad \operatorname{span}\{Y, x_0\} \text{ is closed.}$$

QED

<u>Exercise 3:</u> Let X, Y be Banach spaces, $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{L}(X,Y)$ and suppose that $\forall x \in X$ $A_n(x) \xrightarrow{w} A(x)$ in Y. Show that $A \in \mathcal{L}(X,Y)$

<u>Proof:</u> Since $\{A_n(x)\}_{n \in \mathbb{N}} \subseteq Y$ is weakly convergent, it is norm bounded. So, we can find $M_x > 0$ s.t

$$\|A_n(x)\| \leq M_x \quad \forall n \in \mathbb{N}.$$

By the Banach-Steinhaus theorem, $\exists$ $\hat{M} > 0$ s.t

$$\|A_n\|_{\mathcal{L}} \leq \hat{M} \quad \forall n \in \mathbb{N}.$$

Recall that

$$\|A(x)\| \leq \liminf_{n \to \infty} \|A_n(x)\| \leq \hat{M} \|x\| \quad \forall x \in X,$$

$$\Rightarrow \|A\|_{\mathcal{L}} \leq \hat{M} \text{ and so } A \in \mathcal{L}(X,Y).$$

QED

Exercise 4: Suppose X, Y are Banach spaces and $C \subseteq \mathcal{L}(X,Y)$. Show that C is equicontinuous iff $\exists$ $M>0$ such that $\|A\|_{\mathcal{L}} \leq M$ $\forall A \in C$.

Proof: $\Rightarrow$ On account of equicontinuity, given $\varepsilon>0$, we can find $\delta>0$ such that

$$\|x\| \leq \delta \Rightarrow \|A(x)\| \leq \varepsilon \quad \forall A \in C,$$

$$\Rightarrow \left\|A\left(\delta \frac{x}{\|x\|}\right)\right\| \leq \varepsilon \quad \forall x \in X, \forall A \in C,$$

$$\Rightarrow \|A(x)\| \leq \frac{\varepsilon}{\delta}\|x\| \quad \forall x \in X, \forall A \in C,$$

$$\Rightarrow \|A\|_{\mathcal{L}} \leq \frac{\varepsilon}{\delta} = M \quad \forall A \in C.$$

$\Leftarrow$ For every $x \in X$, we have

$$\|A(x)\| \leq M\|x\|$$

Hence given $\varepsilon>0$, choose $\delta = \frac{\varepsilon}{M} > 0$. Then

$$\|x\| \leq \delta \Rightarrow \|A(x)\| \leq \varepsilon \quad \forall A \in C$$

$$\Rightarrow C = \text{equicontinuous}.$$

QED

Exercise 5: Let $C \subseteq L^p[0,1]$ $1<p<\infty$ and assume that it is bounded. Show that C is uniformly integrable.

Proof: Recall that $L^p[0,1] \hookrightarrow L^1[0,1]$ continuously.

So C is L^1-bounded.

Let $M>0$ be such that $\|u\|_p \leq M$ $\forall u \in C$. Then given $\varepsilon>0$, let $\delta = \left(\frac{\varepsilon}{M}\right)^{p'}$ Then

$$|A| \leq \delta \Rightarrow \int_A |u|\,dx \leq \|u\|_p \|\chi_A\|_{p'} \leq M|A|^{1/p'} \leq \varepsilon,$$

$\Rightarrow$ C is uniformly integrable.

QED

Exercise 6: Consider the sequence $\{f_n(x) = \chi_{[n,n+1]}(x)\}_{n\in\mathbb{N}} \subseteq L^1(\mathbb{R})$. Is this sequence uniformly integrable? Explain

Proof: Let $\lambda>0$. Then

$$\sup_{n\in\mathbb{N}} \int_{\{|f_n|\geq\lambda\}} |f_n(x)|\,dx = 1 \not\to 0$$

So, $\{f_n\}_{n\in\mathbb{N}}$ can not be uniformly integrable

QED

Exercise 7: Let X, Y be Banach spaces and $A \in \mathcal{L}(X,Y)$. Show that

"A is injective $\Leftrightarrow$ $A^*(Y^*) \subseteq X^*$ is w^*-dense"

Proof: $\Rightarrow$ Arguing by contradiction, suppose that $A^*(Y^*)$ is not w^*-dense in X^*. Then we can find $x \in X\setminus\{0\}$ s.t

$$\langle A^*(y^*), x\rangle = 0 \quad \forall y^* \in Y^*,$$

$$\Rightarrow \langle y^*, A(x)\rangle = 0 \quad \forall y^* \in Y^*,$$

$\Rightarrow A(x)=0$ and so $x=0$ (since $A(\cdot)$ is injective)

a contradiction.

$\Leftarrow$ Again we argue by contradiction. So, suppose that $A(\cdot)$ is not injective. Then we can find $x\neq 0$ s.t $A(x)=0$

We have

$$\langle y^*, A(x)\rangle = 0 \quad \forall y^*\in Y^*,$$

$$\Rightarrow \langle A^*(y^*), x\rangle = 0 \quad \forall y^*\in Y^*,$$

$$\Rightarrow \quad x=0 \quad (\text{since by hypothesis } A^*(Y^*) \text{ is } w^*\text{-dense})$$

a contradiction.

QED

Exercise 8: Let X be a Banach space and $A: X\to X^*$ is a linear operator such that $\langle A(u), u\rangle \geq 0 \ \forall u\in X$. Show that $A\in \mathcal{L}(X, X^*)$

Proof: By the Closed Graph Theorem it suffices to show that $\mathrm{Gr}A \subseteq X\times X^*$ is closed. So let $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ be such that

$$u_n \to u \text{ in } X \text{ and } A(u_n)\to x^* \text{ in } X^*$$

We have

$$\langle A(u_n)-A(x), u_n - x\rangle = \langle A(u_n - x), u_n - x\rangle \geq 0 \quad \forall n\in\mathbb{N},\ \forall x\in X$$

$$\Rightarrow \langle x^* - A(x), u - x\rangle \geq 0 \qquad (*)$$

In (*) we choose $x = u + \lambda h$ with $\lambda > 0$, $h \in X$. Then

$$\lambda \langle x^* - A(u), h \rangle - \lambda^2 \langle A(h), h \rangle \geq 0,$$

$$\Rightarrow \langle x^* - A(u), h \rangle - \lambda \langle A(h), h \rangle \geq 0 \quad \forall h \in X, \forall \lambda > 0.$$

Let $\lambda \to 0^+$ to obtain

$$\langle x^* - A(u), h \rangle \geq 0 \quad \forall h \in X,$$

$$\Rightarrow x^* = A(u),$$

$$\Rightarrow \operatorname{Gr} A \subseteq X \times X^* \text{ is closed},$$

$$\Rightarrow A \in \mathcal{L}(X, X^*).$$

QED

Exercise 9: Let X, Y be Banach spaces with X nonreflexive and Y reflexive and suppose that $A \in \mathcal{L}(X,Y)$ is injective. Show that $A(X) \subseteq Y$ is <u>not</u> closed.

Proof: Arguing by contradiction, suppose that $V = A(X)$ is closed. Since $V \subseteq Y$, V is reflexive. Then by Banach's theorem $A: X \to V$ is an isomorphism. So, X is reflexive, a contradiction

QED

Exercise 10: Let $\Omega \subseteq \mathbb{R}^N$ be bounded open, $\{f_n\}_{n \in \mathbb{N}} \subseteq L^1(\Omega)$, $f_n \xrightarrow{w} f$ in $L^1(\Omega)$, $f_n \xrightarrow{a.e} f$. Show that

$$\|f_n - f\|_1 \to 0.$$

<u>Proof:</u> By the Dunford-Pettis theorem $\{f_n\}_{n\geq 1} \subseteq L^1(\Omega)$ is uniformly integrable. Apply Vitali's theorem.

QED

<u>Exercise 11:</u> Let $C = \{u \in L^1[0,1] : u(x) \geq 1 \text{ a.e.}\}$ Is this set w-closed? Explain.

<u>Proof:</u> YES. The set is convex. So, by Mazur's lemma if we show that C is closed, then it will be w-closed. So, let $\{u_n\}_{n\in\mathbb{N}} \subseteq C$ and assume that $u_n \to u$. Then by passing to a subsequence if necessary, we may also say that $u_n(x) \to u(x)$ a.e. Hence

$$u(x) \geq 1 \text{ for a.a } x \in \Omega, \; u \in L^1[0,1]$$

$$\Rightarrow u \in C. \text{ and so } C \text{ is w-closed.}$$

QED

<u>Exercise 12:</u> Suppose $\{u_n, u\}_{n\in\mathbb{N}} \subseteq L^1[0,1]$ and $u_n(x) \to u(x)$ a.e. Is it true that $u_n \to u$ in $L^1[0,1]$? Explain.

<u>Proof:</u> NO Consider the sequence

$$u_n(x) := n\chi_{[0,\frac{1}{n}]}(x) \qquad x \in [0,1], \; n \in \mathbb{N}$$

Then $u_n(x) \to 0$ for a.a $x \in [0,1]$. On the other hand

$$\|u_n\|_1 = 1 \quad \forall \, n \in \mathbb{N},$$

$$\Rightarrow u_n \not\to 0 \text{ in } L^1[0,1].$$

QED

<u>Proposition</u>: If X, Y are topological spaces, X is compact, $f: X \to Y$ is bijective, continuous, <u>then</u> $f(\cdot)$ is homeomorphism.

<u>Exercise 13</u>: Let X be a Banach space and $A \in \mathcal{L}_c(X)$. Show that
"$x_n \xrightarrow{w} x$ in $X \Rightarrow A(x_n) \to A(x)$ in X."

<u>Proof:</u> We have $\{x_n\}_{n \in \mathbb{N}} \subseteq \overline{B}_r$ for some $r > 0$. Since $A \in \mathcal{L}_c(X)$ it follows that

$$\overline{A(\overline{B}_r)} = C \subseteq X \text{ compact.}$$

The map $id: C \to (C, w)$ is bijective, continuous. So, by the Proposition, $id(\cdot)$ is a homeomorphism. Hence on C the norm and weak topologies coincide. Recall that $A \in \mathcal{L}(X_w)$. Therefore $A(x_n) \xrightarrow{w} A(x) \Rightarrow A(x_n) \to A(x)$.

<u>QED</u>

<u>Exercise 14</u>: Let X, Y be Banach spaces, $A: X \to Y$ linear operator and
$$x_n \to 0 \text{ in } X \Rightarrow \{A(x_n)\}_{n \in \mathbb{N}} \subseteq Y \text{ is bounded.}$$
Can we say that $A \in \mathcal{L}(X, Y)$? Explain.

<u>Proof:</u> YES. We show this using a contradiction argument. So, suppose that $A(\cdot)$ is not continuous. Then we can find

$\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that

$$u_n \longrightarrow 0 \quad \text{and} \quad \|A(u_n)\| \geq \varepsilon > 0 \quad \forall n \in \mathbb{N}.$$

We have

$$\left\| A\left(\frac{u_n}{\|u_n\|^{1/2}}\right) \right\| \geq \frac{\varepsilon}{\|u_n\|^{1/2}} \qquad (*)$$

Let $h_n = \dfrac{u_n}{\|u_n\|^{1/2}}$ $n \in \mathbb{N}$. Then $\|h_n\| \longrightarrow 0$ and so by hypothesis

$$\|A(h_n)\| \leq M \quad \text{for some } M>0, \text{ all } n \in \mathbb{N}.$$

$$\Rightarrow \frac{\varepsilon}{\|u_n\|^{1/2}} \leq M \quad \forall n \in \mathbb{N}, \text{ a contradiction.}$$

QED

Exercise 15: Suppose X, Y are Banach spaces, $\{A_n, A\}_{n\in\mathbb{N}} \subseteq \mathcal{L}(X,Y)$ and $\langle y^*, A_n(x)\rangle \to \langle y^*, A(x)\rangle$ $\forall y^* \in Y^*$, $\forall x \in X$.
Show that $\exists M>0$ such that $\|A_n\|_{\mathcal{L}} \leq M$.

Proof: Evidently $\forall x \in X$ $\{A_n(x)\}_{n\in\mathbb{N}} \subseteq Y$ is weakly convergent. So $\exists c(x)>0$ such that

$$\|A_n(x)\| \leq c(x) \quad \forall n \in \mathbb{N},$$

$$\Rightarrow \|A_n\|_{\mathcal{L}} \leq M \quad \forall n \in \mathbb{N}, \text{ with } M>0$$

(uniform boundedness principle).

QED

Exercise 16: Let H be a Hilbert space and $A \in \mathcal{L}(H)$. Suppose that

$$u_n \xrightarrow{w} u \text{ in } H \Rightarrow \|A(u_n)\| \to \|A(u)\|.$$

Show that $A \in \mathcal{L}_c(H)$.

Proof: Since $A \in \mathcal{L}(H) \Rightarrow A \in \mathcal{L}(H_w)$ and so

$$u_n \xrightarrow{w} u \Rightarrow A(u_n) \xrightarrow{w} A(u) \text{ in } H \tag{*}$$

Also by hypothesis

$$u_n \xrightarrow{w} u \Rightarrow \|A(u_n)\| \to \|A(u)\| \tag{**}$$

From (*) and (**) and the Kadec-Klee property of Hilbert spaces, we have

$$u_n \xrightarrow{w} u \Rightarrow A(u_n) \longrightarrow A(u),$$

$\Rightarrow A(\cdot)$ is completely continuous,

$\Rightarrow A \in \mathcal{L}_c(H)$

QED

Exercise 17: Let X, Y be Banach spaces, X is reflexive and $A \in \mathcal{L}(X,Y)$

Show that $A(\overline{B}_1^X) \subseteq Y$ is closed

Proof: The set $A(\overline{B}_1^X)$ is convex.

Also $A \in \mathcal{L}(X_w, Y_w)$ and $\overline{B}_1^X$ is w-compact (since X is reflexive). Hence $A(\overline{B}_1^X)$ is w-compact and so w-closed. Finally the Mazur lemma implies that $A(\overline{B}_1^X)$ is closed

QED

<u>**Exercise 18:**</u> Suppose X, Y are Banach spaces, $\dim X < +\infty$ and $A: X \to Y$ a linear operator. Show that $A \in \mathcal{L}(X,Y)$.

<u>**Proof:**</u> Let $|u| = \|u\|_X + \|A(u)\|_Y$ $\forall u \in X$. This is a norm in X and since X is finite dimensional, it is equivalent to $\|\cdot\|_X$. So, we can find $c > 0$ s.t

$$|u| \leqslant c\|u\|_X \quad \forall u \in X,$$

$$\Rightarrow \|A(u)\|_Y \leqslant c\|u\|_X \quad \forall u \in X,$$

$$\Rightarrow A \in \mathcal{L}(X,Y).$$

<u>QED</u>

<u>**Exercise 19:**</u> Let X be a separable Banach space. Show that $X^*_{w^*}$ is separable

<u>**Proof:**</u> Since X is separable, we know that $(\overline{B}_1^{X^*}, w^*)$ is compact, metrizable, hence separable. But

$$X^* = \bigcup_{n \in \mathbb{N}} n\overline{B}_1^{X^*}$$

Therefore $X^*_{w^*}$ is separable.

<u>QED</u>

<u>**Exercise 20:**</u> Let H be a Hilbert space and $P \in \mathcal{L}(H)$ an orthogonal projection. Show that

$$0 \leqslant P \leqslant I = id.$$

<u>Proof:</u> Let $u \in H$. Then

$$(u - P(u), u) = \|u\|^2 - (P(u), u) \geqslant \|u\|^2 - \|P(u)\| \, \|u\|$$

$$\geqslant \|u\|^2 - \|P\|_{\mathcal{L}} \|u\|^2 = 0$$

$\Rightarrow$ $u \longrightarrow (I - P)(u)$ is positive, hence $P \leq I = \mathrm{id}$.

Recall the direct sum orthogonal decomposition

$$H = V \oplus V^{\perp} \quad \text{with} \quad V = P(H).$$

So, if $u \in H$, then

$$u = \bar{u} + \hat{u} \quad \text{with } \bar{u} \in V, \ \hat{u} \in V^{\perp} \text{ unique}$$

We have

$$(P(u), u) = (P(u), \bar{u} + \hat{u}) = (\bar{u}, \bar{u}) + (\bar{u}, \hat{u})$$

$$= \|\bar{u}\|^2 \geqslant 0.$$

$\Rightarrow$ $P \geqslant 0$.

<u>QED</u>