SOLUTIONS PS 1

PROBLEM 1: By the Open Map Theorem we can find $\varepsilon > 0$ such that

$$\varepsilon B_1^Y \subseteq A(B_1^X)$$

Therefore for every $y \in Y$ with $\|y\| = \delta < \varepsilon$ we can find $x \in B_1^X$ such that $A(x) = y$. Let $M = \frac{1}{\delta}$. Then this is the desired $M > 0$.

QED

PROBLEM 2: Consider a Cauchy sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq X$. Then

$$\|A(u_n) - A(u_m)\| = \|u_n - u_m\| \quad \forall n, m \in \mathbb{N},$$

$$\Rightarrow \{A(u_n)\}_{n \in \mathbb{N}} \subseteq Y \text{ Cauchy},$$

$$\Rightarrow A(u_n) \longrightarrow y \quad (\text{since } Y \text{ is Banach.}$$

Note that A^{-1} isometry thus continuous

Hence

$$u_n \longrightarrow A^{-1}(y),$$

$$\Rightarrow X = \text{Banach.}$$

QED

PROBLEM 3: Let $|u| = \|u\| + \|A(u)\|$. This is a norm on X and so $|\cdot|$, $\|\cdot\|$ are equivalent (see Ch. 3, p. 202). So

$$|u| \leq c\|u\| \text{ for some } c > 0, \text{ all } u \in X$$

$$\Rightarrow \|A(u)\| \leq c\|u\| \text{ and so } A \in \mathcal{L}(X, Y).$$ QED

PROBLEM 4: Suppose $\operatorname{int} C \neq \emptyset$ and let $x \in \operatorname{int} C$. We can find $r > 0$ such that

$$\overline{B}_r(x) \subseteq \operatorname{int} C \subseteq C,$$

$$\Rightarrow \overline{B}_r \text{ is compact and so } \dim X < \infty.$$

QED

PROBLEM 5: Let $u \in \operatorname{int} V$. We can find $r > 0$ such that

$$B_r(u) \subseteq V$$

We have $B_r(0) = B_r(u) - u \subseteq V$. Given $x \in X \setminus \{0\}$ we have

$$\frac{r}{2} \frac{v}{\|v\|} \in B_r(0) \subseteq V.$$

$$\Rightarrow v \in V \text{ and so } V = X.$$

QED