1 Basic Topology

Topology, as its name suggests[1], deals with geometric properties of objects that depend only on their relative positions and not on notions such as size or magnitude. The properties studied by topology are preserved by certain continuous transformations. Discontinuous transformations destroy topological properties. In this chapter we present the basic items of point-set topology that are needed to examine certain topics of applied analysis. We do not claim to have an exhaustive presentation of the subject.

1.1 Basic Notions

We start with the definition of topology.

Definition 1.1.1. Let X be a set and let $\tau \subseteq 2^X$ be such that the following hold:
(a) X and $\emptyset$ both belong to τ;
(b) τ is closed under arbitrary unions, that is, if $\{U_i\}_{i\in I} \subseteq \tau$ is any family of sets in τ, then $\bigcup_{i\in I} U_i \in \tau$;
(c) τ is closed under finite intersections, that is, if $\{U_i\}_{i\in I} \subseteq \tau$ is a finite family of sets in τ, then $\bigcap_{i\in I} U_i \in \tau$.
Then we say that τ is a **topology** on X. The sets in τ are called **open sets**. The complements of the elements of τ are called **closed sets**. In addition we say that the pair (X, τ) is a **topological space**.

Remark 1.1.2. When the topology τ is clearly understood from the context, then we drop it and simply say that X is a topological space. From the definition above it is clear that the family of closed sets contains X and $\emptyset$ and it is closed under finite unions and arbitrary intersections. If X is a set with two topologies τ_1 and τ_2 such that $\tau_1 \subseteq \tau_2$, then we say that τ_1 is **weaker** than τ_2 or that τ_2 is **stronger** than τ_1. The intersection of any family of topologies on X is also a topology that is weaker than every member of the family but stronger than any other topology with this property. Note that for any set X there is a **strongest topology** on X, namely $\tau = 2^X$ known as the **discrete topology**. Moreover, there also exists a **weakest topology** on X, namely $\tau = \{X, \emptyset\}$ known as the **trivial topology**.

In general, a topology is a very large collection of subsets. So it is useful to have a smaller collection of elements of τ, which generates the topology by taking unions.

Definition 1.1.3. Let (X, τ) be a topological space. A **basis** (or **base**) for the topology τ is a subfamily $\mathcal{B}$ of τ such that every member of τ is the union of elements in $\mathcal{B}$. The

1 it comes from the Greek word $\tau\acute{o}\pi o\varsigma$ = location or position

https://doi.org/10.1515/9783110532982-001

elements of $\mathcal{B}$ are called **basic open sets** and τ is the topology **generated** by $\mathcal{B}$. A subfamily $\mathcal{L}$ of τ is a **subbasis** of the topology τ if the family of finite intersections of elements in $\mathcal{L}$ is a basis for τ. The elements of $\mathcal{L}$ are called **subbasic open sets**.

In the definition above, we have assumed a topology on X and defined a basis for it. On the other hand, one might start with a basis and using it, generates a topology on X by taking unions. However, not every family in 2^X is a basis for a topology. The next proposition gives necessary and sufficient conditions for a family to generate a topology.

Proposition 1.1.4. *A family $\mathcal{B} \subseteq 2^X$ is a basis for a topology on X if and only if*
(a) $\bigcup \mathcal{B} = X$, *that is, the union of the elements of $\mathcal{B}$ is X;*
(b) *if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.*

Proof. $\Longrightarrow$: The assertion in (a) follows from the fact that X is open; see Definition 1.1.3. Let us prove (b). We know that $B_1 \cap B_2$ is open. So, according to Definition 1.1.3, $B_1 \cap B_2$ is the union of elements in $\mathcal{B}$. Hence we can find $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

$\Longleftarrow$: Let τ be all unions of elements of $\mathcal{B}$. We need to show that τ is a topology on X; see Definition 1.1.1. Evidently $\emptyset \in \tau$ and $X \in \tau$; see (a). In addition, from its definition, τ is closed under arbitrary unions. We have to show that τ is closed under finite intersections. So, let $U_1, U_2 \in \tau$. Then $U_1 \cap U_2 \in \tau$. Given $x \in U_1 \cap U_2$, there exist $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. Therefore, $x \in B_1 \cap B_2 \subseteq U_1 \cap U_2$. By (b) there is $B(x) \in \mathcal{B}$ such that $x \in B(x) \subseteq U_1 \cap U_2$. Obviously, $U_1 \cap U_2 = \bigcup_x B_x \in \tau$. Thus τ is a topology on X. ☐

Remark 1.1.5. We say that τ is the topology generated by $\mathcal{B}$ and we often write $\tau(\mathcal{B})$ to emphasize the basis generating the topology.

Corollary 1.1.6. *If (X, τ) is a topological space and $\mathcal{B}$ is a subfamily of τ such that for each $U \in \tau$ and $x \in U$, we can find $V \in \mathcal{B}$ such that $x \in V \subseteq U$, then $\mathcal{B}$ is a basis for the topology τ.*

Proposition 1.1.7. *If (X, τ) is a topological space and $\mathcal{B}$ is a basis for τ, then $U \in \tau$, that is, U is open, if and only if for every $x \in U$ there exists $V_x \in \mathcal{B}$ such that $x \in V_x \subseteq U$.*

Proof. $\Longrightarrow$: This follows from (b) of Proposition 1.1.4.

$\Longleftarrow$: We have $U = \bigcup_x V_x \in \tau$. ☐

Definition 1.1.8. Two bases $\mathcal{B}$ and $\mathcal{B}'$ of X are said to be **equivalent** if $\tau(\mathcal{B}) = \tau(\mathcal{B}')$.

Directly from Propositions 1.1.4 and 1.1.7 we have the following characterization of equivalent topological bases.

Proposition 1.1.9. *Two bases $\mathcal{B}$ and $\mathcal{B}'$ in X are equivalent if and only if*
(a) *for every $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$;*
(b) *for every $B' \in \mathcal{B}'$ and $x \in B'$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq \mathcal{B}'$.*

Example 1.1.10. In $\mathbb{R}^N$ with $N \in \mathbb{N}$, let $\mathcal{B} = \left\{B_r(x) : x \in \mathbb{R}^N, r > 0\right\}$ with $B_r(x) = \{u \in \mathbb{R}^N : |u - x| < r\}$. Then $\mathcal{B}$ is a basis for the so-called **Euclidean topology** (or **standard topology**) on $\mathbb{R}^N$. So, every open set in $\mathbb{R}^N$ is the union of open balls. More generally this is also true for every metric space.

There is a local version of the notion of topological basis.

Definition 1.1.11. Let (X, τ) be a topological space and $x \in X$. We say that $\mathcal{B}(x) \subseteq \tau$ is a **local basis** (or a **local base**) at x if the following hold:
(a) $x \in V$ for every $V \in \mathcal{B}(x)$;
(b) if $x \in U \in \tau$, then there exists $V \in \mathcal{B}(x)$ such that $x \in V \subseteq U$.

Definition 1.1.12. Let (X, τ) be a topological space and $A \subseteq X$.
(a) A **neighborhood** of $x \in X$ is any open set U such that $x \in U$.
(b) We say that $x \in A$ is an **interior point** of A if we can find $U \in \tau$ such that $x \in U \subseteq A$. The **interior** of A, denoted by $\operatorname{int} A$ (or by $\mathring{A}$), is the set of all interior points of A.
(c) We say that $x \in X$ is a **cluster point** (or a **limit point** or an **accumulation point**) of A if every open set containing x contains a point of A distinct from x. The set of all cluster points of A is called the **derived set** of A and is denoted by A'. The **closure** of A, denoted by $\overline{A}$ (or $\operatorname{cl} A$), is the union of A with its set of cluster points, that is, $\overline{A} = A \cup A'$.
(d) We say that $x \in X$ is a **boundary point** of A if $x \in \overline{A} \cap (\overline{X \setminus A})$. The set of boundary points of A is called the **boundary** of A and is denoted by $\operatorname{bd} A$ (or by ∂A).

Remark 1.1.13. Note that a cluster point or a boundary point of A need not belong to A. In the sequel we denote by $\mathcal{N}(x)$ the family of all neighborhoods of $x \in X$.

Proposition 1.1.14. *If (X, τ) is a topological space and $A, C \subseteq X$, then the following hold:*
(a) $\operatorname{int} A = \bigcup\{U \in \tau : U \subseteq A\}$*, that is,* $\operatorname{int} A$ *is the largest open set contained in* A*;*
(b) A *is open if and only if* $A = \operatorname{int} A$*;*
(c) $A \subseteq C$ *implies* $\operatorname{int} A \subseteq \operatorname{int} C$*;*
(d) $\operatorname{int}(A \cap C) = \operatorname{int} A \cap \operatorname{int} C$.

Proof. (a) Let $\tilde{A} = \bigcup\{U \in \tau : U \subseteq A\}$. Then $\tilde{A}$ is open and by Definition 1.1.12(b) it is clear that $\operatorname{int} A \subseteq \tilde{A}$. On the other hand, if $x \in \tilde{A}$, then there is $U \in \tau$, $U \subseteq A$ such that $x \in U$. Hence, x is an interior point of A, therefore $\tilde{A} \subseteq \operatorname{int} A$. We conclude that $\tilde{A} = \operatorname{int} A$.

(b) This is an immediate consequence of (a).

(c) We have $\operatorname{int} A \subseteq A \subseteq C$ and since $\operatorname{int} A$ is open, it follows that $\operatorname{int} A \subseteq \operatorname{int} C$, see part (a).

(d) We have $A \cap C \subseteq A$ and $A \cap C \subseteq C$. Then $\operatorname{int}(A \cap C) \subseteq \operatorname{int} A$ and $\operatorname{int}(A \cap C) \subseteq \operatorname{int} C$ because of part (c). This gives

$$\operatorname{int}(A \cap C) \subseteq \operatorname{int} A \cap \operatorname{int} C\,. \tag{1.1.1}$$

On the other hand, $\operatorname{int} A \cap \operatorname{int} C$ is an open subset of $A \cap C$. Hence, because of (a),

$$\operatorname{int} A \cap \operatorname{int} C \subseteq \operatorname{int}(A \cap C) . \tag{1.1.2}$$

From (1.1.1) and (1.1.2) we conclude that $\operatorname{int} A \cap \operatorname{int} C = \operatorname{int}(A \cap C)$. □

Remark 1.1.15. In general it is not true that $\operatorname{int}(A \cup C) = \operatorname{int} A \cup \operatorname{int} C$. Indeed let $X = \mathbb{R}$ with the Euclidean topology, see Example 1.1.10, and let $A = [0, 1]$ and $C = [1, 2]$. Then

$$\operatorname{int} A = (0, 1) , \quad \operatorname{int} C = (1, 2) \quad \text{and} \quad \operatorname{int}(A \cup C) = (0, 2) .$$

In general we can easily show that if $\{A_i\}_{i\in I}$ is an arbitrary family of subsets of X, then

$$\bigcup_{i\in I} \operatorname{int} A_i \subseteq \operatorname{int} \bigcup_{i\in I} A_i .$$

There is an analogous proposition for the closure.

Proposition 1.1.16. *If (X, τ) is a topological space and $A, C \subseteq X$, then the following hold:*
(a) $\overline{A} = \bigcap\{D : D$ *closed*, $D \supseteq A\}$, *that is, $\overline{A}$ is the smallest closed set containing A;*
(b) *A is closed if and only if $A = \overline{A}$;*
(c) *$A \subseteq C$ implies $\overline{A} \subseteq \overline{C}$;*
(d) $\overline{A \cup C} = \overline{A} \cup \overline{C}$.

Proof. (a) Let $A^* = \bigcap\{D : D \text{ closed}, D \supseteq A\}$. Evidently, A^* is closed and so $X \setminus A^*$ is open. Hence, if $x \notin A^*$, then we find $U \in \mathcal{N}(x)$ such that $U \cap A = \emptyset$. Therefore, $x \notin (A \cup A') = \overline{A}$ and so $\overline{A} \subseteq A^*$. Now suppose that $x \in A^* \setminus \overline{A}$. Then there exists $U \in \mathcal{N}(x)$ such that $U \cap A = \emptyset$. Let $C = X \setminus U$. Then C is closed and $C \supseteq A$. Hence $A^* \subseteq C$ and so $x \in C$, a contradiction. Therefore $\overline{A} = A^*$.

(b) This is an immediate consequence of (a).

(c) We have $A \subseteq C \subseteq \overline{C}$ and since $\overline{C}$ is closed, it follows that $\overline{A} \subseteq \overline{C}$, see part (a).

(d) Note that $\overline{A} \cup \overline{C}$ is closed and contains $A \cup C$. Hence

$$\overline{A \cup C} \subseteq \overline{A} \cup \overline{C} . \tag{1.1.3}$$

Since $A, C \subseteq A \cup C$, we have $\overline{A}, \overline{C} \subseteq \overline{A \cup C}$, see part (c). Hence

$$\overline{A} \cup \overline{C} \subseteq \overline{A \cup C} . \tag{1.1.4}$$

From (1.1.3) and (1.1.4) we conclude that $\overline{A \cup C} = \overline{A} \cup \overline{C}$. □

Remark 1.1.17. In general it is not true that $\overline{A \cap C} = \overline{A} \cap \overline{C}$. To see this, let $X = \mathbb{R}$ with the Euclidean topology and let $A = (0, 1)$ as well as $C = (1, 2)$. Then $\overline{A \cap C} = \emptyset$ and $\overline{A} \cap \overline{C} = [0, 1] \cap [1, 2] = \{1\}$. In general we can easily show that if $\{A_i\}_{i\in I}$ is an arbitrary family of subsets of X, then

$$\overline{\bigcap_{i\in I} A_i} \subseteq \bigcap_{i\in I} \overline{A_i} .$$

In addition, the following formulas are easy to verify:

– $x \in A'$ if and only if $x \in \overline{(A' \setminus \{x\})}$;
– $(A \cup C)' = A' \cup C'$, $A' \setminus C' \subseteq (A \setminus C)'$, $A'' \subseteq A'$;
– $\left(\bigcap_{i \in I} A_i\right)' \subseteq \bigcap_{i \in I} A_i'$ with an arbitrary index set I;
– $\bigcup_{i \in I} A_i' \subseteq \left(\bigcup_{i \in I} A_i\right)'$ with an arbitrary index set I;
– $\overline{A'} = A'$;
– $A \subseteq C$ implies $A' \subseteq C'$;
– $(A \setminus \{x\})' = A' = (A \cup \{x\})'$.

The last formula means that the derived set remains unchanged if we add or remove a finite number of elements. If $x \in A \setminus A'$, then we say that x is **isolated**.

Proposition 1.1.18. *If (X, τ) is a topological space and $A \subseteq X$, then the following hold:*
(a) $\operatorname{bd} A = \overline{A} \cap \overline{(X \setminus A)} = \operatorname{bd}(X \setminus A)$;
(b) $\operatorname{bd} A$, $\operatorname{int} A$, $\operatorname{int}(X \setminus A)$ *are pairwise disjoint sets whose union is X;*
(c) $\operatorname{bd} A$ *is a closed set;*
(d) $\overline{A} = \operatorname{int} A \cup \operatorname{bd} A$;
(e) *A is open if and only if* $\operatorname{bd} A \subseteq X \setminus A$;
(f) *A is closed if and only if* $\operatorname{bd} A \subseteq A$;
(g) *A is closed and open (usually called **clopen**) if and only if* $\operatorname{bd} A = \emptyset$.

Proof. (a)–(d) These are immediate consequences of Definition 1.1.12.

(e) $\Longrightarrow$: Since A is open we have $A = \operatorname{int} A$ due to Proposition 1.1.14(b). From part (b) we know that $\operatorname{int} A$ and $\operatorname{bd} A$ are disjoint sets. Therefore $\operatorname{bd} A \subseteq X \setminus A$.

$\Longleftarrow$: Since $\operatorname{bd} A \subseteq X \setminus A$, no point of A is a boundary point. Hence, every point of A is an interior point, see part (d). Therefore, $A = \operatorname{int} A$, that is, A is open.

(f) This follows from (e) by taking complements.

(g) Combine (e) and (f). □

Definition 1.1.19. A subset A of a topological space X is said to be **dense** if $\overline{A} = X$. We say that the topological space X is **separable** if it has a countable, dense subset.

Remark 1.1.20. It is easy to see that A is dense in the topological space (X, τ) if and only if for every $U \in \tau$, $U \neq \emptyset$ we have $U \cap A \neq \emptyset$. Clearly $\mathbb{R}^N$ is separable since we can take the set of vectors with rational coordinates as a countable, dense set.

Definition 1.1.21. A subset A of a topological space X is said to be **nowhere dense** if $\operatorname{int} \overline{A} = \emptyset$.

Remark 1.1.22. From the definition above we see that $A \subseteq X$ is nowhere dense if and only if $X \setminus \overline{A}$ is dense in X. It follows that $A \subseteq X$ is nowhere dense if and only if $X \setminus \overline{(X \setminus \overline{A})} = \emptyset$ or that A is nowhere dense if and only if $A \subseteq \overline{(X \setminus \overline{A})}$. Any set A that contains a dense set is itself dense. Similarly, any subset of a nowhere dense set is nowhere dense. The closure of a nowhere dense set is nowhere dense.

Proposition 1.1.23. *If X is a topological space and $A \subseteq X$ is open or closed, then* $\operatorname{bd} A$ *is nowhere dense.*

Proof. Suppose that A is open. Then $\operatorname{bd} A = \overline{A} \setminus A$, see Proposition 1.1.18(d). Hence, $\operatorname{int} \operatorname{bd} A = \operatorname{int}(\overline{A} \setminus A) = \emptyset$, which shows that $\operatorname{bd} A$ is nowhere dense.

Similarly, if A is closed, then $\operatorname{bd} A = A \cap \overline{(X \setminus A)}$, see Definition 1.1.12(d). Therefore, by Proposition 1.1.14(d), $\operatorname{int} \operatorname{bd} A = \operatorname{int} A \cap \operatorname{int} \overline{(X \setminus A)}$. Hence, $\operatorname{int} \operatorname{bd} A = \emptyset$ and so $\operatorname{bd} A$ is nowhere dense in X. □

Definition 1.1.24. Let (X, τ) be a topological space and $A \subseteq X$. The **subspace or relative topology** on A is the family

$$\tau(A) = \{U \cap A : U \in \tau\} .$$

It is also called the **trace** of τ on A. It is easy to see that $\tau(A)$ is a topology on A.

Proposition 1.1.25. *If (X, τ) is a topological space, $\mathcal{B}$ is a basis for the topology τ and $A \subseteq X$, then $\mathcal{B}(A) = \{U \cap A : U \in \mathcal{B}\}$ is a basis for $\tau(A)$.*

Proof. Let $U \in \tau$ and $u \in U \cap A$. We can find $V \in \mathcal{B}$ such that $u \in V \subseteq U$. Then $u \in V \cap A \subseteq U \cap A$. This implies that $\mathcal{B}(A)$ is a basis for $\tau(A)$; see Corollary 1.1.6. □

Proposition 1.1.26. *If (X, τ) is a topological space, $A \in \tau$ and $V \in \tau(A)$, then $V \in \tau$.*

Proof. Since $V \in \tau(A)$ we have $V = U \cap A$ with $U \in \tau$. But $U \cap A \in \tau$ since $A \in \tau$. □

Proposition 1.1.27. *If (X, τ) is a topological space and $A \subseteq X$, then $D \subseteq A$ is $\tau(A)$-closed if and only if $D = C \cap A$ with closed $C \subseteq X$.*

Proof. $\Longrightarrow$: Since $D \subseteq A$ is $\tau(A)$-closed, that is, relatively closed, we have $A \setminus D = U \cap A$ with $U \in \tau$. Then $D = A \setminus (A \setminus D) = A \setminus (U \cap A) = (X \setminus U) \cap A = C \cap A$ with closed $C = X \setminus U$.

$\Longleftarrow$: Let $U = X \setminus C$. Then $U \in \tau$ and we have

$$A \setminus D = A \setminus (C \cap A) = (X \setminus C) \cap A = U \cap A ,$$

which implies that $A \setminus D$ is $\tau(A)$-open and so D is $\tau(A)$-closed. □

As a consequence of Proposition 1.1.26 we have the following observation concerning neighborhoods of a point $x \in A$.

Corollary 1.1.28. *If (X, τ) is a topological space, $A \subseteq X$, $x \in A$ and $V \subseteq A$, then $V \in \mathcal{N}_A(x)$, where $\mathcal{N}_A(x)$ denotes the $\tau(A)$-neighborhoods of x, if and only if $V = U \cap A$ with $U \in \mathcal{N}(x)$.*

This discussion on relativization of topologies leads naturally to the following notion, which will be used in the sequel.

Definition 1.1.29. A property of topological spaces is said to be **hereditary** if every subset with the relative (subspace) topology exhibits this property.

The notion of continuity is central in point-set topology. It is the main tool that allows us to determine which mathematical properties are intrinsic to a particular topological space.

Definition 1.1.30. Let X, Y be topological spaces. We say that a map $f\colon X \to Y$ is **continuous** at $x \in X$ if for every $U \in \mathcal{N}(f(x))$ we can find $V \in \mathcal{N}(x)$ such that $f(V) \subseteq U$. We say that $f\colon X \to Y$ is **continuous** if it is continuous at every $x \in X$.

Remark 1.1.31. From the last definition it is clear that continuity is a local property. The next proposition provides a useful global characterization of continuity.

Proposition 1.1.32. *If (X, τ_X) and (Y, τ_Y) are two topological spaces and $f\colon X \to Y$, then f is continuous if and only if $f^{-1}(\tau_Y) \subseteq \tau_X$, that is, f returns open sets in Y to open sets in X.*

Proof. $\Longrightarrow$: Let $U \in \tau_Y$. Then U is a neighborhood of each of its points. So, $f^{-1}(U)$ contains a neighborhood of everyone of its points. Hence $f^{-1}(U) \in \tau_X$.

$\Longleftarrow$: This is immediate from Definition 1.1.30. □

Remark 1.1.33. Since f^{-1} preserves all set theoretic operations, in the proposition above we may replace τ_Y by a basis $\mathcal{B}_Y$ or even better by a subbasis $\mathcal{L}_Y$.

We have a counterpart of Proposition 1.1.32 with closed sets instead of open sets.

Proposition 1.1.34. *If X and Y are topological spaces and $f\colon X \to Y$, then f is continuous if and only if for every closed $C \subseteq Y$, $f^{-1}(C)$ is closed in X.*

Proposition 1.1.35. *If X and Y are topological spaces and $f\colon X \to Y$, then the following statements are equivalent.*

(a) *f is continuous;*
(b) *$f(\overline{A}) \subseteq \overline{f(A)}$ for every $A \subseteq X$;*
(c) *$\overline{f^{-1}(C)} \subseteq f^{-1}(\overline{C})$ for every $C \subseteq Y$.*

Proof. (a) $\Longrightarrow$ (b): Let $A \subseteq X$ and $x \in \overline{A}$. Consider $U \in \mathcal{N}(f(x))$ and choose $V \in \mathcal{N}(x)$ such that $f(V) \subseteq U$, see Definition 1.1.30. We have

$$\begin{aligned} x \in \overline{A} \quad &\Longrightarrow \quad V \cap A \neq \emptyset \quad \Longrightarrow \quad f(V \cap A) \neq \emptyset \\ &\Longrightarrow \quad f(V) \cap f(A) \neq \emptyset \quad \Longrightarrow \quad U \cap f(A) \neq \emptyset\,. \end{aligned}$$

Since $U \in \mathcal{N}(f(x))$ is arbitrary it follows that $x \in \overline{f(A)}$. Hence $f(\overline{A}) \subseteq \overline{f(A)}$.

(b) $\Longrightarrow$ (c): Let $A = f^{-1}(C)$. Then by hypothesis $f(\overline{A}) \subseteq \overline{f(A)} = \overline{f(f^{-1}(C))} \subseteq \overline{C}$ and so $\overline{A} = \overline{f^{-1}(C)} \subseteq f^{-1}(\overline{C})$.

(c) $\Longrightarrow$ (a): Let $C \subseteq Y$ be closed. Then by hypothesis $\overline{f^{-1}(C)} \subseteq f^{-1}(C)$ and so $\overline{f^{-1}(C)} = f^{-1}(C)$, that is, $f^{-1}(C)$ is closed. From Proposition 1.1.34 it follows that f is continuous. □

Proposition 1.1.36. *Let X, Y and Z be topological spaces.*
(a) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous maps, then $g \circ f\colon X \to Z$ is continuous.*
(b) *If $f\colon X \to Y$ is a continuous map and $A \subseteq X$, then $f|_A\colon A \to Y$ is continuous for the subspace topology of A.*
(c) *If $X = \bigcup_{i \in I} U_i$ with U_i open and $f\colon X \to Y$ is a map such that $f|_{U_i}$ is continuous, then $f\colon X \to Y$ is continuous.*

Proof. (a) If U is open in Z, then $g^{-1}(U)$ is open in Y and $f^{-1}(g^{-1}(U))$ is open in X, see Proposition 1.1.32. But recall that $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$. So, by Proposition 1.1.32, $g \circ f$ is continuous.

(b) Let $i\colon A \to X$ be the inclusion map where A is endowed with the subspace topology. Evidently i is continuous and since $f|_A = f \circ i$ we derive the conclusion using part (a).

(c) Let $V \subseteq Y$ be open. Then $f^{-1}(V) \cap U_i = \left(f|_{U_i}\right)^{-1}(V)$ is open in X for all $i \in I$. Therefore $f^{-1}(V) = \bigcup_{i \in I} f^{-1}(V) \cap U_i$ is open in X. Taking Proposition 1.1.32 into account yields the continuity of f. □

Continuing in the same way, we prove the so-called "Pasting Lemma."

Proposition 1.1.37 (Pasting Lemma). *If X and Y are topological spaces, $X = A \cup B$ with closed subsets A and B of X, $f\colon A \to Y$ and $g\colon B \to Y$ are continuous maps where A and B are endowed with the subspace topology and $f(x) = g(x)$ for all $x \in A \cap B$. Then $h\colon X \to Y$ defined by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases},$$

is continuous.

Proof. Let C be a closed subset of Y. Then

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)\,. \tag{1.1.5}$$

By hypothesis $f^{-1}(C)$ is closed in A and since A is closed in Y, from Proposition 1.1.27, we have that $f^{-1}(C)$ is closed in X. Similarly $g^{-1}(C)$ is closed in X. From (1.1.5) it follows that $h^{-1}(C)$ is closed in X. Hence, by Proposition 1.1.34, h is continuous. □

In general the direct image of an open (resp. closed) set by a map need not be open (resp. closed) even if the map is continuous. For this reason we introduce the following definition.

Definition 1.1.38. Let X and Y be two topological spaces. We say that a map $f\colon X \to Y$ is **open** (respectively, **closed**) if the image of every open (respectively, closed) set in X is open (respectively, closed) in Y.

Remark 1.1.39. It is easy to see that the notions of continuous map, open map, and closed map are independent.

Proposition 1.1.40. *Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f\colon X \to Y$, then the following statements are equivalent:*
(a) *f is open;*
(b) *$f(\operatorname{int} A) \subseteq \operatorname{int} f(A)$ for every $A \subseteq X$;*
(c) *if $\mathcal{B}_X$ is a basis for τ_X, then $f(\mathcal{B}_X) \subseteq \tau_Y$.*

Proof. (a) $\Longrightarrow$ (b): We have $f(\operatorname{int} A) \subseteq f(A)$ and by hypothesis $f(\operatorname{int} A)$ is open. By Proposition 1.1.14(a) it follows that $f(\operatorname{int} A) \subseteq \operatorname{int} f(A)$.

(b) $\Longrightarrow$ (c): Let $V \in \mathcal{B}_X$. Then by hypothesis $f(V) = f(\operatorname{int} V) \subseteq \operatorname{int} f(V)$. Hence, $f(V) = \operatorname{int} f(V)$, that is, $f(V) \in \tau_Y$.

(c) $\Longrightarrow$ (a): Let $V \subseteq X$ be open. Then $V = \bigcup_{i \in I} V_i$ with $V_i \in \mathcal{B}_X$. We have

$$f(V) = f\left(\bigcup_{i \in I} V_i\right) = \bigcup_{i \in I} f(V_i) \in \tau_Y .$$

Therefore, f is open. □

Next we identify a subfamily of continuous functions that is in the core of point-set topology.

Definition 1.1.41. Let X and Y be two topological spaces and $f\colon X \to Y$ is a bijection. We say that f is a **homeomorphism** if both f and f^{-1} are continuous. Then we say that the spaces X and Y are **homeomorphic**. Instead of homeomorphism we also say that f is **bicontinuous**.

As an easy consequence of this definition and of Proposition 1.1.40 we have the following proposition.

Proposition 1.1.42. *Let X and Y be topological spaces and let $f\colon X \to Y$ be a bijection, then the following statements are equivalent:*
(a) *f is a homeomorphism;*
(b) *f is continuous and open;*
(c) *f is continuous and closed;*
(d) *$f(\overline{A}) = \overline{f(A)}$ for every $A \subseteq X$.*

Remark 1.1.43. Given a homeomorphism $f\colon X \to Y$, $U \subseteq X$ is open if and only if $f(U) \subseteq Y$ is open. Thus a homeomorphism gives a bijection between the topologies of X and Y. Hence, any property of X that is expressed using only the topology of X, yields the same property on Y. Such a property of X is said to be a **topological property** of X.

1.2 Separation and Countability Properties – Convergence

The so-called separation properties determine how rich the supply is of open sets in a given topological space. This is important because the supply of open sets determines the supply of continuous functions. We need to have a rich enough supply of continuous functions in order to produce interesting results.

We start with a notion, which for analysis, is the minimal requirement for a topological space.

Definition 1.2.1. A topological space X is said to be **Hausdorff** (or T_2**-space**) if for every pair $x, u \in X$ we can find $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(u)$ such that $U \cap V = \emptyset$.

Since our aim is to use topology to investigate problems in analysis, from now on all topological spaces considered are Hausdorff. Let us give an example of a space that is important in algebraic geometry and that is not Hausdorff.

Example 1.2.2. Let $n \in \mathbb{N}$ and let $\mathcal{P}$ denote the set of all polynomials in n variables $\{x_1, \dots, x_n\}$. Given $p \in \mathcal{P}$, let

$$Z(p) = \{(x_1, \dots, x_n) \in \mathbb{R}^n : p(x_1, \dots, x_n) = 0\} .$$

Let $\mathcal{B}$ be the family of all complements of the set $Z(p)$ with $p \in \mathcal{P}$. One can show that $\mathcal{B}$ is a basis for a topology of $\mathbb{R}^n$. This topology is called the "Zariski topology" on $\mathbb{R}^n$ and it turns out that it is not Hausdorff.

Proposition 1.2.3. *The Hausdorff property is hereditary and topological.*

Proof. Let (X, τ) be the topological space and $A \subseteq X$ endowed with the subspace topology $\tau(A)$. Consider two distinct points $x, u \in A$. We can find $U, V \in \tau$ with $x \in U$ and $u \in V$ such that $U \cap V = \emptyset$. Then $U \cap A \in \tau(A)$, $V \cap A \in \tau(A)$ and $(U \cap A) \cap (V \cap A) = \emptyset$. Hence, $(A, \tau(A))$ is Hausdorff.

Let X be a Hausdorff topological space, Y a topological space, and $f : X \to Y$ a homeomorphism. If $y, v \in Y$ are distinct points, then $f^{-1}(y), f^{-1}(v) \in X$ are distinct as well. Since X is Hausdorff we can find $U, V \in \tau$ such that $f^{-1}(y) \in U$, $f^{-1}(v) \in V$ and $U \cap V = \emptyset$. This implies that $y \in f(U)$, $v \in f(V)$ are both open sets in Y and $f(U) \cap f(V) = \emptyset$. Therefore, Y is Hausdorff as well. □

Proposition 1.2.4. *If X is a Hausdorff topological space and $A \subseteq X$ is finite, then A is closed.*

Proof. It suffices to show that every singleton $\{x\}$ is closed. So let $u \in X$ with $u \neq x$. Then we can find $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(u)$ such that $U \cap V = \emptyset$. This means that $x \notin \overline{\{u\}}$. Therefore $\overline{\{x\}} = \{x\}$ and so every singleton $\{x\}$ is closed. □

Proposition 1.2.5. *If X is a Hausdorff topological space and $A \subseteq X$, then $x \in A'$, that is, x is a cluster point of A, if and only if every $U \in \mathcal{N}(x)$ contains infinitely many points of A.*

Proof. $\Longrightarrow$: Arguing by contradiction, suppose that we can find $U \in \mathcal{N}(x)$ such that $U \cap A$ is a finite set. Then $U \cap (A \setminus \{x\})$ is finite. Let $U \cap (A \setminus \{x\}) = \{x_k\}_{k=1}^n$. From Proposition 1.2.4 we know that $\{x_k\}_{k=1}^n$ is a closed subset of X. Hence $X \setminus \{x_k\}_{k=1}^n$ is open. Then

$$V = U \cap \left(X \setminus \{x_k\}_{k=1}^n\right) \in \mathcal{N}(x)$$

and $V \cap A = \emptyset$, a contradiction to the fact that $x \in A'$.

$\Longleftarrow$: By hypothesis, every $U \in \mathcal{N}(x)$ intersects A at infinitely many points. Then according to Definition 1.1.12(c), we have $x \in A'$. □

Proposition 1.2.6. *For a topological space X the following statements are equivalent:*
(a) *X is Hausdorff;*
(b) *Given $x \in X$ and $u \neq x$ we can find $U \in \mathcal{N}(x)$ such that $u \notin \overline{U}$;*
(c) *For every $x \in X$ we have $\{x\} = \bigcap\{\overline{U} \colon U \in \mathcal{N}(x)\}$.*

Proof. (a) $\Longrightarrow$ (b): Let $x \in X$ and $u \neq x$. Since by hypothesis X is Hausdorff we can find $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(u)$ such that $U \cap V = \emptyset$. This means that $u \notin \overline{U}$.

(b) $\Longrightarrow$ (c): Let $u \neq x$. By hypothesis we can find $U \in \mathcal{N}(x)$ such that $u \notin \overline{U}$. Therefore we conclude that $\{x\} = \bigcap\{\overline{U} \colon U \in \mathcal{N}(x)\}$.

(c) $\Longrightarrow$ (a): Let $x \neq u$. We can find $U \in \mathcal{N}(x)$ such that $u \notin \overline{U}$ and $V \in \mathcal{N}(u)$ such that $x \notin \overline{V}$. We set $U' = U \cap (X \setminus \overline{V}) \in \mathcal{N}(x)$ and $V' = V \cap (X \setminus \overline{U}) \in \mathcal{N}(u)$. Evidently $U' \cap V' = \emptyset$ and this shows that X is Hausdorff. □

Now we strengthen the separation property.

Definition 1.2.7. A Hausdorff topological space X is said to be **regular** (or T_3**-space**) if for each closed set $C \subseteq X$ and each $x \notin C$ we can find open sets U and V such that $x \in U$, $C \subseteq V$ and $U \cap V = \emptyset$.

Proposition 1.2.8. *A Hausdorff topological space X is regular if and only if for every point $x \in X$ and every $U \in \mathcal{N}(x)$ we can find $W \in \mathcal{N}(x)$ such that $\overline{W} \subseteq U$.*

Proof. $\Longrightarrow$: Let $x \in X$ and $U \in \mathcal{N}(x)$. Then $X \setminus U$ is a closed set not containing x. Since by hypothesis X is regular, we can find open sets W, V such that

$$x \in W\,, \quad X \setminus U \subseteq V \quad \text{and} \quad W \cap V = \emptyset\,. \tag{1.2.1}$$

We have $W \subseteq X \setminus V$ and so $\overline{W} \subseteq X \setminus V$ since $X \setminus V$ is closed. Then, because of (1.2.1),

$$\overline{W} \subseteq X \setminus V \subseteq X \setminus (X \setminus U) = U\,.$$

This means that $W \in \mathcal{N}(x)$ is the desired neighborhood of x.

$\Longleftarrow$: Let $x \in X$ and let $C \subseteq X$ be closed such that $x \notin C$. Then $X \setminus C \in \mathcal{N}(x)$ and so by hypothesis we can find $W \in \mathcal{N}(x)$ such that $\overline{W} \subseteq X \setminus C$. Then W and $X \setminus \overline{W}$ are open sets such that $x \in W$, $C \subseteq X \setminus \overline{W}$ and $W \cap (X \setminus \overline{W}) = \emptyset$ which by Definition 1.2.7 means that X is regular. □

Proposition 1.2.9. *A Hausdorff topological space X is regular if and only if for every point $x \in X$ and every closed set $C \subseteq X$ such that $x \notin C$ we can find open sets U, V for which we have $\overline{U} \cap \overline{V} = \emptyset$.*

Proof. $\Longrightarrow$: Let $x \in X$ and let $C \subseteq X$ be a closed set such that $x \notin C$. Since by hypothesis, X is regular, invoking Proposition 1.2.8, we can find $W \in \mathcal{N}(x)$ such that $\overline{W} \subseteq X \setminus C$. A new application of Proposition 1.2.8 produces $U \in \mathcal{N}(x)$ such that $\overline{U} \subseteq W$. Let $V = X \setminus \overline{W}$, which is open. Then we obtain $\overline{U} \subseteq W \subseteq \overline{W} \subseteq X \setminus C$, which gives $C \subseteq X \setminus \overline{W} = V$. Therefore, U and V is the desired pair of open sets.

$\Longleftarrow$: This is obvious from Definition 1.2.7. □

Proposition 1.2.10. *The regularity property is hereditary and topological.*

Proof. Let $A \subseteq X$ and let $D \subseteq A$ be relatively closed and let $x \in A \setminus D$. From Proposition 1.1.27 we have $D = C \cap A$ with closed $C \subseteq X$. Since $x \notin C$ and X is regular, we can find open subsets U, V of X such that $x \in U$, $C \subseteq V$ and $U \cap V = \emptyset$. Then $U \cap A$, $V \cap A$ are relatively open in A, $x \in U \cap A$ and $D \subseteq V \cap A$. This shows that A with the relative (subspace) topology is regular.

Let $f\colon X \to Y$ be a homeomorphism and $y \in Y$, $C \subseteq Y$ closed with $y \notin C$. Let $x = f^{-1}(y)$ and $\hat{C} = f^{-1}(C)$. Evidently $\hat{C} \subseteq X$ is closed and $x \notin \hat{C}$. Since X is regular we can find open subsets $\hat{U}$, $\hat{V}$ of X such that $x \in \hat{U}$, $\hat{C} \subseteq \hat{V}$ and $\hat{U} \cap \hat{V} = \emptyset$. This gives $y \in f(\hat{U}) = U$, $f(\hat{C}) = C \subseteq f(\hat{V}) = V$ and $f(\hat{U}) \cap f(\hat{V}) = \emptyset$ since f is a homeomorphism. But from Proposition 1.1.42 we have that U, V are open subsets of Y. Hence we conclude that Y is regular. □

We further strengthen the separation property.

Definition 1.2.11. A Hausdorff topological space X is said to be **normal** (or **T_4-space**) if for each pair A, C of disjoint closed sets in X, we can find open sets U, V such that $A \subseteq U$, $C \subseteq V$ and $U \cap V = \emptyset$.

Remark 1.2.12. The definition above can be equivalently stated as follows: "If U_1, U_2 are open sets in X such that $X = U_1 \cup U_2$, then we can find closed subsets C_1, C_2 of X such that $C_1 \subseteq U_1$, $C_2 \subseteq U_2$ and $X = C_1 \cup C_2$."

The next two propositions characterize normality and are proven with arguments similar to the ones used in Propositions 1.2.8 and 1.2.9.

Proposition 1.2.13. *A Hausdorff topological space X is normal if and only if for each closed set $C \subseteq X$ and each open set $U \subseteq X$ such that $C \subseteq U$ we can find an open set $V \subseteq X$ for which we have $C \subseteq V \subseteq \overline{V} \subseteq U$.*

Proposition 1.2.14. *A Hausdorff topological space X is normal if and only if for each pair A, C of disjoint closed sets in X we can find open sets U, V in X such that $A \subseteq U$, $C \subseteq V$ and $\overline{U} \cap \overline{V} = \emptyset$.*

Proposition 1.2.15. (a) *A closed subset of a normal space is normal.*
(b) *Normality is preserved under continuous, closed surjections.*

Proof. (a) Let X be a normal topological space and $A \subseteq X$ a closed set. Suppose that $C \subseteq A$ is relatively closed. Then $C \subseteq X$ is closed by Proposition 1.1.27. This observation leads immediately to the normality of A.

(b) Let X be a normal topological space, Y a topological space, and $f\colon X \to Y$ a continuous, closed surjection. Suppose that U_1, U_2 are open subsets of Y such that

$Y = U_1 \cup U_2$. Then $\hat{U}_1 = f^{-1}(U_1)$, $\hat{U}_2 = f^{-1}(U_2)$ are open in X and $X = \hat{U}_1 \cup \hat{U}_2$. The normality of X implies that we can find closed subsets $\hat{C}_1, \hat{C}_2$ of X such that $\hat{C}_1 \subseteq \hat{U}_1$, $\hat{C}_2 \subseteq \hat{U}_2$ and $X = \hat{C}_1 \cup \hat{C}_2$; see Remark 1.2.12.

Since f is closed we have that $C_1 = f(\hat{C}_1)$, $C_2 = f(\hat{C}_2)$ are closed subsets of Y and $C_1 \subseteq U_1$, $C_2 \subseteq U_2$ as well as $Y = C_1 \cup C_2$. According to Remark 1.2.12, this means that Y is normal as well. ☐

Remark 1.2.16. Part (a) of Proposition 1.2.15 fails if the subset is not closed. For a counterexample we refer to Dugundji [91, p. 145].

As we already mentioned in the beginning of this section, richness in open sets implies richness in continuous functions. This is illustrated in the theorem that follows. The result is known as "Urysohn's Lemma."

Theorem 1.2.17 (Urysohn's Lemma). *A Hausdorff topological space X is normal if and only if for each pair A, C of disjoint closed subsets of X we can find a continuous function $f\colon X \to [0,1]$ such that $f\big|_A = 0$ and $f\big|_C = 1$.*

Proof. $\Longrightarrow$: Let D be the set of all rationals r of the form $r = k/2^n$ with $0 \le k/2^n \le 1$, that is, $k = 0, 1, \ldots, 2^n$ dyadic fractions. We show that for every $r \in D$ we can assign an open set $U(r)$ such that

(a) $A \subseteq U(0) \subseteq \overline{U(0)} \subseteq X \setminus C, \quad U(1) = X \setminus C$.

(b) $r < r'$ implies $\overline{U(r)} \subseteq U(r')$.

We proceed by induction on the exponent $n \in \mathbb{N}$. So, let

$$E_n = \left\{U\left(\frac{k}{2^n}\right) : k = 0, 1, \ldots, 2^n\right\}, n \in \mathbb{N}\,.$$

Then $E_0 = \{U(0), U(1) = X \setminus C\}$ and (a) is satisfied by Proposition 1.2.13. Suppose that E_{n-1} have been constructed. Clearly we need to define $U(k/2^n)$ for $k =$ odd. For $k =$ odd, from the induction hypothesis, we have

$$\overline{U\left(\frac{k-1}{2^n}\right)} \subseteq U\left(\frac{k+1}{2^n}\right),$$

see (b). So we define $U(k/2^n) = U$ with U being an open set such that, due to Proposition 1.2.13,

$$\overline{U\left(\frac{k-1}{2^n}\right)} \subseteq U \subseteq \overline{U} \subseteq U\left(\frac{k+1}{2^n}\right).$$

This completes the induction and we have defined the collection

$$\left\{U\left(\frac{k}{2^n}\right) : k = 0, 1, \ldots, 2^n, n \in \mathbb{N}\right\}.$$

We define the desired function f by setting

$$f(x) = \begin{cases} 0 & \text{if } x \in U(r) \text{ for every } r = \text{dyadic fraction as above}\,, \\ \sup\{r\colon x \notin U(r)\} & \text{otherwise}\,. \end{cases}$$

Then f has values in $[0, 1]$ and $f|_A = 0, f|_C = 1$. So it remains to show that f is continuous. Note that the intervals $\{[0, a), (a, 1] : 0 < a < 1\}$ form a subbasis for $[0, 1]$ with the Euclidean topology. So, according to Remark 1.1.33 it suffices to show that $f^{-1}([0, a))$ and $f^{-1}((a, 1])$ are open. Note that $f(x) < a$ if and only if $x \in U(r)$ for some $r < a$. It follows that $f^{-1}([0, a)) = \bigcup_{r<a} U(r)$, which is open. Similarly, $f(x) > a$ if and only if $x \notin \overline{U(r)}$ for some $r > a$. Therefore $f^{-1}((a, 1]) = \bigcup_{r>a}(X \setminus \overline{U(r)})$, which is open. This proves the continuity of f.

$\Longleftarrow$: Let $A, C \subseteq X$ be disjoint closed sets. By hypothesis we can find a continuous function $f : X \to [0, 1]$ such that

$$f|_A = 0 \quad \text{and} \quad f|_C = 1 . \tag{1.2.2}$$

Let $U = \{x \in X : f(x) < 1/2\}$ and $V = \{x \in X : f(x) > 1/2\}$. Then $U, V \subseteq X$ are open, $U \cap V = \emptyset$, $A \subseteq U$, $C \subseteq V$, see (1.2.2), which implies that X is normal. □

Remark 1.2.18. We can have a form of this result that is a little more flexible. To be more precise, we can replace $[0, 1]$ by $[a, b]$ with $a, b \in \mathbb{R}$, $a \leq b$ and $f|_A = a, f|_C = b$. Indeed, let f_0 be the continuous separating function postulated by Theorem 1.2.17. Then set $f = (b - a)f_0 + a$. Evidently this function has the desired properties.

There is another such functional characterization of normality, namely the so-called "Tietze Extension Theorem." We state this result at the end of this section and for its proof, which is rather technical, we refer to Dugundji [91].

Evidently we have

$$\text{Normal} \quad \Longrightarrow \quad \text{Regular} \quad \Longrightarrow \quad \text{Hausdorff} .$$

None of these implications is in general reversible. Between regular and normal spaces we can fit another class given in the next definition.

Definition 1.2.19. A Hausdorff topological space X is said to be **completely regular** if for each $x \in X$ and each closed set $C \subseteq X$ with $x \notin C$, we can find a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f|_C = 1$.

Now we pass to the countability properties of a topological space.

Definition 1.2.20. (a) A topological space X is said to be **first countable** if it has a countable local basis at each point of X.
(b) A topological space X is said to be **second countable** if it has a countable basis.

Remark 1.2.21. Evidently a second countable space is also first countable. The converse is not true. Every metric space (X, d) is first countable. Indeed for every $x \in X$, $\mathcal{B}(x) = \{B_r(x) : r \in \mathbb{Q}\}$ with $B_r(x) = \{u \in X : d(u, x) < r\}$ is a countable local basis at $+x$ and so X is first countable.

Proposition 1.2.22. *Every second countable space is separable.*

Proof. Let X be a second countable space and let $\mathcal{B}$ be the countable basis of X. Let D be the countable set formed by choosing an element from each nonempty basic open set. Then Corollary 1.1.6 implies that $\overline{D} = X$. □

Remark 1.2.23. The converse of the proposition above is not true. Consider the space $X = \mathbb{R}$ topologized with the topology that has as its basis intervals of the form $(a, b]$ with $a, b \in \mathbb{R}$. This topology is known as the **upper limit topology** and is denoted by τ_u. We can easily check that the Euclidean topology on $X = \mathbb{R}$ is weaker than τ_u. The space $(\mathbb{R}, \tau_u)$ is first countable. To see this, consider $\mathcal{B}(x) = \{(r, x] : r \in \mathbb{Q}\}$ for each $x \in \mathbb{R}$.

In addition, $(\mathbb{R}, \tau_u)$ is separable. Indeed, the rationals are a countable dense subset. However, $(\mathbb{R}, \tau_u)$ is not second countable. To see this, note that if $\{(a_n, b_n]\}_{n\in\mathbb{N}}$ is a countable collection in τ_u, then by choosing $a, b \neq b_n$ for all $n \in \mathbb{N}$, the open set $(a, b]$ cannot be expressed as a union of sets in the countable collection. The proposition above also says that every nonseparable metric space is first countable but not second countable.

Proposition 1.2.24. (a) *Second countability is preserved by continuous open surjections.*
(b) *Second countability is hereditary.*
(c) *Separability is preserved by continuous surjections.*

Proof. (a) Let X be a second countable topological space, Y another topological space, and $f : X \to Y$ a continuous open surjection. Consider a basis $\{U_n\}_{n\in\mathbb{N}}$ for the topology of X, and using Corollary 1.1.6, we see that $\{f(U_n)\}_{n\in\mathbb{N}}$ is a countable basis for Y.

(b) This is obvious.

(c) Let X be a separable topological space, Y another topological space and $f : X \to Y$ a continuous surjection. Consider $D \subseteq X$ as being a countable dense subset. From Proposition 1.1.35(b) we have $Y = f(X) = f(\overline{D}) \subseteq \overline{f(D)}$. Hence, $Y = \overline{f(D)}$ and $f(D)$ is countable. □

Remark 1.2.25. Clearly, an open subset of a separable topological space is separable for the subspace topology. If X is a second countable topological space, then every subset of X endowed with the subspace topology is separable.

Definition 1.2.26. Let (X, τ) be a topological space.
(a) An **open cover** of X is a collection $\mathcal{D} \subseteq \tau$ such that $X = \bigcup\{U : U \in \mathcal{D}\}$. A **subcover** of an open cover $\mathcal{D}$ is a subfamily $\mathcal{D}'$ of $\mathcal{D}$ such that $X = \bigcup\{U : U \in \mathcal{D}'\}$.
(b) We say that X is a **Lindelöf space** if every open cover contains a countable subcover.

The next result relates the Lindelöf property with second countability. It is known as "Lindelöf's Theorem."

Theorem 1.2.27 (Lindelöf's Theorem). *Every second countable space is Lindelöf.*

Proof. Let X be a second countable topological space and $\{U_n\}_{n\geq1}$ a countable basis of X. Consider an open cover $\mathcal{D} = \{V_i\}_{i\in I}$ of X. For each $x \in X$, let $V_{i(x)} \in \{V_i\}_{i\in I}$ be such that $x \in V_{i(x)}$. Let $U_{n(x)} \in \{U_n\}_{n\geq1}$ be such that $x \in U_{n(x)} \subseteq V_{i(x)}$. Then the family

$\{U_{n(x)}\}_{x\in X}$ is a countable open cover of X. For each $U_{n(x)}$ let $V'_{i(x)} \in \mathcal{D}$ be such that $U_{n(x)} \subseteq V'_{i(x)}$. Then the collection $\{V'_{i(x)}\}_{x\in X}$ is a countable subcover of $\mathcal{D}$. Therefore, X is Lindelöf. □

Remark 1.2.28. The converse of the Theorem above is not true. Consider the space $(\mathbb{R}, \tau_u)$; see Remark 1.2.23. Then we can show that it is Lindelöf (see Dugundji [91]), but it is not second countable; see again Remark 1.2.23.

Proposition 1.2.29. (a) *The Lindelöf property is preserved by continuous surjections.*
(b) *A closed subset of a Lindelöf space is Lindelöf for the subspace topology.*

Proof. (a) Let X be a Lindelöf space, Y another topological space, and $f\colon X \to Y$ a continuous surjection. Consider an open cover $\{U_i\}_{i\in I}$ of Y. Then $\{V_i\}_{i\in I} = \{f^{-1}(U_i)\}_{i\in I}$ is an open cover of X. Since X is Lindelöf, we can find a countable subcover $\{V_n\}_{n\in\mathbb{N}} = \{f^{-1}(U_n)\}_{n\in\mathbb{N}}$. Then $\{U_n\}_{n\in\mathbb{N}}$ is a countable subcover of $\{U_i\}_{i\in I}$ and so we conclude that Y is Lindelöf.

(b) Let X be a Lindelöf space and $C \subseteq X$ a closed subset. Consider an open cover $\{V_i\}_{i\in I}$ of C with the subspace topology. Then $V_i = U_i \cap C$ with $U_i \subseteq X$ open. Then $\{U_i, X \setminus C\}_{i\in I}$ is an open cover of X. Since X is Lindelöf we can find a countable subcover $\{U_n\}_{n\in\mathbb{N}}$. Then $\{U_n \cap C\}_{n\in\mathbb{N}}$ is a countable subcover of $\{V_i\}_{i\in I}$. So, we conclude that C with the subspace topology is Lindelöf. □

We know that a sequence is a map from $\mathbb{N}$ into X but it is more convenient to think of a sequence as a subset of X indexed by $\mathbb{N}$. We generalize this notion by replacing $\mathbb{N}$ with a more general index set.

Definition 1.2.30. Let X be a set.
(a) A **relation** is any subset $R \subseteq X \times X$. Given a relation, it is more suggestive to write xRy instead of $(x, y) \in R$. We say that R is **reflexive** if xRx for all $x \in X$. We say that R is **symmetric** if xRy implies yRx. We say that R is **antisymmetric** if xRy and yRx imply $x = y$. We say that R is **transitive** if xRy and yRz imply xRz.
(b) A relation R is called an **equivalence relation** if it is reflexive, symmetric, and transitive.
(c) A relation R is called a **partial order** if it is antisymmetric and transitive. In this case we write $x \le y$ if and only if xRy or $x = y$ (a reflexive partial order) and $x < y$ if and only if xRy and $x \ne y$ (a strict partial order). A **linear order** R is a partial order such that for all $x, u \in X$, either xRu or uRx. A **chain** is a linearly ordered subset of a partially ordered set.
(d) A **directed set** is a partially ordered set $(I, \le)$ such that for any $\alpha, \beta \in I$ we can find $k \in I$ such that $\alpha \le k$ and $\beta \le k$.

Remark 1.2.31. Many authors require that a partial order is also reflexive. Definition 1.2.30(c) is more flexible and allows both "≤" and "<" as partial orders. For any set V let $X = 2^V$ be the collection of all subsets of V. We write $A \le C$ if and only if $C \subseteq A$ for $C, A \in 2^V$. This is a partial order, the reverse ordering of the sets.

Definition 1.2.32. Let X be a set. A **net** in X is a map $x\colon D \to X$ with a directed set D. The directed set D is known as the **index set** of the net.

Remark 1.2.33. As for sequences, we denote the map $x\colon D \to X$ simply by $\{x_\alpha\}_{\alpha\in D}$.

Definition 1.2.34. Let (X, τ) be a topological space. We say that a net $\{x_\alpha\}_{\alpha\in I}$ converges to some $x \in X$ if for every $U \in \mathcal{N}(x)$ there exists $\alpha_0 = \alpha_0(U)$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$, that is, $\{x_\alpha\}$ is eventually in every neighborhood of x. We say that x is the **limit** of the net $\{x_\alpha\}_{\alpha\in I}$ and we write $x_\alpha \to x$ or $x_\alpha \xrightarrow{\tau} x$ if we want to emphasize the topology τ.

Proposition 1.2.35. *A topological space X is Hausdorff if and only if every convergent net has a unique limit.*

Proof. $\Longrightarrow$: Arguing by contradiction, suppose that for a net $\{x_\alpha\}_{\alpha\in I} \subseteq X$ we have

$$x_\alpha \to x \quad \text{and} \quad x_\alpha \to \hat{x} \quad \text{with } x \neq \hat{x}\,.$$

Due to the Hausdorff property we can find $U \in \mathcal{N}(x)$ and $\hat{U} \in \mathcal{N}(\hat{x})$ such that $U \cap \hat{U} = \emptyset$. Furthermore we can find $\alpha_0, \hat{\alpha}_0 \in I$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$ and $x_\alpha \in \hat{U}$ for all $\alpha \geq \hat{\alpha}_0$. Since I is a directed set we can find $\alpha_* \in I$ such that $\alpha_* \geq \alpha_0, \alpha_* \geq \hat{\alpha}_0$. Then, $x_\alpha \in U$ and $x_\alpha \in \hat{U}$ for all $\alpha \geq \alpha_*$, a contradiction since $U \cap \hat{U} = \emptyset$. This proves the uniqueness of the limit.

$\Longleftarrow$: We argue again indirectly. To this end, suppose that X is not Hausdorff. Then we can find $x, u \in X$ with $x \neq u$ such that for every $U \in \mathcal{N}(x)$ and every $V \in \mathcal{N}(u)$ there holds $U \cap V \neq \emptyset$. For each $(U, V) \in \mathcal{N}(x) \times \mathcal{N}(u)$, let $x_{UV} \in U \cap V$ and note that the net $\{x_{UV}\}$ converges to both x and u, contradicting our hypothesis. □

Proposition 1.2.36. *If X is a topological space and $A \subseteq X$, then $x \in \overline{A}$ if and only if we can find a net $\{x_\alpha\}_{\alpha\in I} \subseteq A$ such that $x_\alpha \to x$.*

Proof. $\Longrightarrow$: If $U \in \mathcal{N}(x)$, then $U \cap A \neq \emptyset$. Let $x_U \in U \cap A$. Then $\{x_U\}_{U\in\mathcal{N}(x)}$ is a net in A with $\mathcal{N}(x)$ ordered by reverse inclusion, that is, $U_1 \leq U_2$ if and only if $U_2 \subseteq U_1$, and $x_U \to x$.

$\Longleftarrow$: This is obvious. □

Proposition 1.2.37. *If X and Y are topological spaces, $x \in X$ and $f\colon X \to Y$, then f is continuous at x if and only if for every net $x_\alpha \to x$ we have $f(x_\alpha) \to f(x)$.*

Proof. $\Longrightarrow$: According to Definition 1.1.30, given $U \in \mathcal{N}(f(x))$, we can find $V \in \mathcal{N}(x)$ such that $f(V) \subseteq U$. If $x_\alpha \to x$, then we can find $\alpha_0 \in I$ such that $x_\alpha \in V$ for all $\alpha \geq \alpha_0$. Then $f(x_\alpha) \in f(V) \subseteq U$ for all $\alpha \geq \alpha_0$. Since $U \in \mathcal{N}(f(x))$ is arbitrary, we conclude that $f(x_\alpha) \to f(x)$.

$\Longleftarrow$: Arguing by contradiction, suppose that f is not continuous at x. Then there is $V \in \mathcal{N}(f(x))$ such that $f^{-1}(V) \notin \mathcal{N}(x)$. Then $x \in \overline{X \setminus f^{-1}(V)}$ and so by Proposition 1.2.36

we can find a net $\{x_\alpha\}_{\alpha\in I} \subseteq X \setminus f^{-1}(V)$ such that $x_\alpha \to x$. By hypothesis we have $f(x_\alpha) \to f(x)$. Since $Y \setminus V$ is closed and $f(x_\alpha) \in Y \setminus V$ for all $\alpha \in I$, it follows that $f(x) \in Y \setminus V$, a contradiction. This proves the continuity of f at x. ☐

The next notion generalizes that of a subsequence.

Definition 1.2.38. Let X be a topological space. A net $\{u_\beta\}_{\beta\in J} \subseteq X$ is a **subnet** of a net $\{x_\alpha\}_{\alpha\in I}$ if there exists a map $\vartheta\colon J \to I$ such that
(a) $u_\beta = x_{\vartheta(\beta)}$ for every $\beta \in J$;
(b) for each $\alpha_0 \in I$, there exists $\beta_0 \in J$ such that $\beta \geq \beta_0$ implies $\vartheta(\beta) \geq \alpha_0$.

Remark 1.2.39. A subsequence of a sequence is a subnet. But we can have subnets of a sequence that are not subsequences. Indeed, let $\{x_n\}_{n\in\mathbb{N}} = \{n^2 + 1\}_{n\in\mathbb{N}}$ and $\{y_{m,n}\}_{(m\times n)\in\mathbb{N}\times\mathbb{N}} = \{m^2 + 2mn + n^2 + 1\}_{(m\times n)\in\mathbb{N}\times\mathbb{N}}$. Then $\{y_{m,n}\}_{(m\times n)\in\mathbb{N}\times\mathbb{N}}$ is a subnet of $\{x_n\}_{n\in\mathbb{N}}$. To see this, let $\vartheta\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be defined by $\vartheta(m, n) = m + n$. However, note that $\{y_{m,n}\}_{(m\times n)\in\mathbb{N}\times\mathbb{N}}$ is not a subsequence of $\{x_n\}_{n\in\mathbb{N}}$.

For the next result, the limits need not be unique.

Proposition 1.2.40. *If X is a topological space, then a net in X converges to a point if and only if every subnet converges in X to the same point.*

Proof. $\Longrightarrow$: This is obvious.

$\Longleftarrow$: Suppose that $\{x_\alpha\}_{\alpha\in I}$ is a net in X and assume that every subnet of $\{x_\alpha\}_{\alpha\in I}$ converges to the same limit x. Arguing by contradiction, suppose that the net $\{x_\alpha\}_{\alpha\in I}$ does not converge to x. Then we can find $U \in \mathcal{N}(x)$ such that for any $\alpha \in I$ we can find $\vartheta(\alpha) \geq \alpha$ such that $x_{\vartheta(\alpha)} \notin V$. If we set $u_\alpha = x_{\vartheta(\alpha)}$, then $\{u_\alpha\}_{\alpha\in I}$ is a subnet of $\{x_\alpha\}_{\alpha\in I}$, which does not converge to x. This contradicts our hypothesis. So, $x_\alpha \to x$. ☐

Remark 1.2.41. For a bounded real net $\{x_\alpha\}_{\alpha\in I}$ we can define the **limit superior** and the **limit inferior** by setting

$$\liminf_\alpha x_\alpha = \sup_{\alpha\in I} \inf_{\vartheta\geq\alpha} x_\vartheta \quad \text{and} \quad \limsup_\alpha x_\alpha = \inf_{\alpha\in I} \sup_{\vartheta\geq\alpha} x_\vartheta\ .$$

Evidently, $x_\alpha \to x$ if and only if $x = \liminf_\alpha x_\alpha = \limsup_\alpha x_\alpha$.

Nets were introduced because in general, sequences are not enough to describe a given topology τ.

Definition 1.2.42. Let (X, τ) be a topological space. We denote by τ_{seq} the topology on X whose closed sets are the sequentially τ-closed sets in X.

Remark 1.2.43. From the definition above, it follows directly that τ_{seq} is the strongest, that is, the largest topology on X for which the converging sequences are the τ-converging sequences. Hence, $\tau \subseteq \tau_{\text{seq}}$ and $\tau = \tau_{\text{seq}}$ if and only if (X, τ) is first countable.

In Theorem 1.2.17 we produced a characterization of normal spaces. We conclude this section by providing an alternative characterization of normality. The result is known as the “Tietze Extension Theorem.” We do not prove it, since later we will prove a more general result known as the “Dugundji Extension Theorem”; see Theorem 1.7.29.

Theorem 1.2.44 (Tietze Extension Theorem). *A Hausdorff topological space X is normal if and only if for every closed $C \subseteq X$ and for every continuous $f\colon C \to \mathbb{R}$ there exists a continuous function $\hat{f}\colon X \to \mathbb{R}$ such that $\hat{f}|_C = f$. Moreover, if $|f(x)| \le M$ for some $M > 0$ and for all $x \in A$, then $\hat{f}$ can be chosen so that $|\hat{f}(x)| \le M$ for all $x \in X$.*

1.3 Weak, Product, and Quotient Topologies

Let X and $\{Y_i\}_{i\in I}$ be topological spaces and $f_i\colon X \to Y_i$ be continuous functions. From Proposition 1.1.32 we see that if we strengthen (enrich) the topology on X, we preserve the continuity of the f_i's. Thus it is natural to inquire what the smallest topology on X is, which preserves the continuity of the f_i's. This leads to the notions of weak and product topologies, which occur in a prominent position in many areas of analysis such as functional analysis.

Definition 1.3.1. Let X be a nonempty set, let $\{(Y_i, \tau_i)\}_{i\in I}$ be a family of Hausdorff topological spaces and let $f_i\colon X \to Y_i$ with $i \in I$ be a family of functions. The **weak topology** or **initial topology** on X generated by the family of functions $\{f_i\}_{i\in I}$ is the weakest topology on X that makes all f_i's continuous. The weak topology is denoted by $\mathrm{w}(X, \{f_i\})$ or simply by w if X and $\{f_i\}$ are clearly understood.

Remark 1.3.2. Simple set theory reveals that the weak topology is generated, that is, it has as subbasis, the sets of the form

$$\left\{f_i^{-1}(V)\colon V \in \tau_i, i \in I\right\} . \tag{1.3.1}$$

Recalling that to check continuity it suffices to consider the inverse image of subbasic sets, another more economical subbasis is given by

$$\left\{f_i^{-1}(V)\colon V \in \mathcal{L}_i, i \in I\right\} \tag{1.3.2}$$

with a subbasis $\mathcal{L}_i$ for the topology τ_i. Then a basis for the weak topology is produced by taking finite intersections of the sets above; see (1.3.1) and (1.3.2). An important special case is when $Y_i = \mathbb{R}$ for all $i \in I$. This is the case of the weak topology in functional analysis. Then the subbasic elements are of the form

$$U(x; f, \varepsilon) = \left\{u \in X\colon |f(u) - f(x)| < \varepsilon\right\}$$

with $x \in X, f \in \{f_i\}$ and $\varepsilon > 0$.

Proposition 1.3.3. *A net $\{x_\alpha\}_{\alpha\in J}$ converges to x for the weak topology, which is denoted by $x_\alpha \xrightarrow{\mathrm{w}} x$, if and only if $f_i(x_\alpha) \to f_i(x)$ for all $i \in I$.*

Proof. $\Longrightarrow$: This follows from Proposition 1.2.37, since each f_i is w-continuous.

$\Longleftarrow$: Let $V = \bigcap_{k=1}^{n} f_{i_k}^{-1}(V_{i_k})$ be a basic neighborhood of X where $V_{i_k} \in \tau_{i_k}$. Since by hypothesis $f_{i_k}(x_\alpha) \to f_{i_k}(x)$, we can find $\alpha_{i_k} \in J$ such that

$$x_\alpha \in f_{i_k}^{-1}(V_{i_k}) \quad \text{for all } \alpha \geq \alpha_{i_k} . \tag{1.3.3}$$

Since J is directed we can find $\alpha_0 \geq \alpha_{i_k}$ for all $k \in \{1, \ldots, n\}$. Then $x_\alpha \in V$ for all $\alpha \geq \alpha_0$ because of (1.3.3). This implies $x_\alpha \xrightarrow{\mathrm{w}} x$ in X. □

Proposition 1.3.4. *If Z is another topological space and $g\colon Z \to X$ is a map, then g is continuous for the weak topology on X if and only if $f_i \circ g$ is continuous for all $i \in I$.*

Proof. $\Longrightarrow$: From Proposition 1.1.36(a) we know that $f_i \circ g$ is continuous for all $i \in I$.

$\Longleftarrow$: Let $U \subseteq X$ be weakly open. Then

$$U = \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} f_i^{-1}(V_i) \quad \text{with } V_i \in \tau_i .$$

This gives

$$g^{-1}(U) = \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} g^{-1}(f_i^{-1}(V_i)) = \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} (f_i \circ g)^{-1}(V_i) ,$$

which is open in Z, and thus g is continuous. □

Consider X endowed with the weak topology $\mathrm{w}(X, \{f_i\})$. Suppose that $A \subseteq X$. Then we can consider on A the subspace topology induced by $\mathrm{w}(X, \{f_i\})$. However, we can also consider the weak topology $\mathrm{w}(A, \{f_i|_A\})$; see Proposition 1.1.36(b). It is natural to ask what the relation is between these two topologies on A. It is easy to see that the two topologies have the same convergent nets. This leads to the next result.

Proposition 1.3.5. *If X is endowed with the weak topology $\mathrm{w}(X, \{f_i\})$ and $A \subseteq X$, then* $\mathrm{w}(X, \{f_i\})|_A = \mathrm{w}(A, \{f_i|_A\})$.

As we already mentioned, an analyst requires that a topological space is at least Hausdorff. So we need to know the conditions that guarantee that the weak topology is Hausdorff.

Definition 1.3.6. Let X and $\{Y_i\}_{i \in I}$ be sets and let $f_i\colon X \to Y_i$ be a family of functions. We say that the family $\{f_i\}_{i \in I}$ is **separating** (or **total**) if for every pair $(x, u) \in X \times X$ with $x \neq u$ we can find $i_0 \in I$ such that $f_{i_0}(x) \neq f_{i_0}(u)$.

Proposition 1.3.7. *If $\mathrm{w}(X, \{f_i\})$ is the weak topology on X, then $\mathrm{w}(X, \{f_i\})$ is Hausdorff if and only if $\{f_i\}_{i \in I}$ is separating.*

Proof. $\Longrightarrow$: Arguing by contradiction, suppose that the family $\{f_i\}_{i \in I}$ is not separating. So, we can find a pair $(x, u) \in X \times X$ with $x \neq u$ such that $f_i(x) = f_i(u)$ for all $i \in I$. Let $U \in \mathcal{N}_{\mathrm{w}}(x)$ where $\mathcal{N}_{\mathrm{w}}(x)$ is the family of weak neighborhoods of x. Then we can find

$\{f_{i_k}\}_{k=1}^n \subseteq \{f_i\}_{i\in I}$ and $V_{i_k} \in \tau_{i_k}$ with $k \in \{1, \dots, n\}$ such that

$$x \in \bigcap_{k=1}^{n} f_{i_k}^{-1}(V_{i_k}) \subseteq U . \tag{1.3.4}$$

Since $f_i(x) = f_i(u)$ for all $i \in I$, we have

$$u \in \bigcap_{k=1}^{n} f_{i_k}^{-1}(V_{i_k}) .$$

Due to (1.3.4) it follows $u \in U$. We infer that (X, w) is not Hausdorff, a contradiction.

$\Longleftarrow$: As before, we proceed indirectly. Suppose that (X, w) is not Hausdorff. Then according to Proposition 1.2.35 we can find a net $\{x_\alpha\}_{\alpha\in I} \subseteq X$ such that

$$x_\alpha \xrightarrow{w} x \quad \text{and} \quad x_\alpha \xrightarrow{w} \hat{x} , \quad x \neq \hat{x} .$$

For every $i \in I$ we have $f_i(x_\alpha) \to f_i(x)$ and $f_i(x_\alpha) \to f_i(\hat{x})$ in Y_i, which is Hausdorff. Hence, $f_i(x) = f_i(\hat{x})$ for all $i \in I$, see Proposition 1.2.35. This means that the family $\{f_i\}_{i\in I}$ is not separating, a contradiction. □

Next we derive some useful results concerning the weak topology. Let (X, τ) be a Hausdorff topological space. We will use the following notations:

- $C(X, \mathbb{R}) = \{f : X \to \mathbb{R} : f \text{ is continuous}\}$;
- $C_b(X, \mathbb{R}) = \{f : X \to \mathbb{R} : f \text{ is bounded and continuous}\}$.

Proposition 1.3.8. *If (X, τ) is a Hausdorff topological space, then* $\mathrm{w}(X, C(X, \mathbb{R})) = \mathrm{w}(X, C_b(X, \mathbb{R}))$.

Proof. Since $C_b(X, \mathbb{R}) \subseteq C(X, \mathbb{R})$ we infer that $\mathrm{w}(X, C_b(X, \mathbb{R})) \subseteq \mathrm{w}(X, C(X, \mathbb{R}))$. So we need to show that the opposite inclusion also holds. Let U be a subbasic open set in $\mathrm{w}(X, C(X, \mathbb{R}))$. Then we have

$$U(x; f, \varepsilon) = \{u \in X : |f(u) - f(x)| < \varepsilon\}$$

with $x \in X, f \in C(X, \mathbb{R})$ and $\varepsilon > 0$. Let

$$g(u) = \min\{f(x) + \varepsilon, \max\{f(x) - \varepsilon, f(u)\}\} .$$

Evidently we have $g \in C_b(X, \mathbb{R})$ and $U(x; g, \varepsilon) = U(x; f, \varepsilon)$, which implies that $\mathrm{w}(X, C(X, \mathbb{R})) \subseteq \mathrm{w}(X, C_b(X, \mathbb{R}))$. This proves the assertion. □

The next theorem characterizes completely regular spaces (see Definition 1.2.19) via the weak topologies of the previous proposition.

Theorem 1.3.9. *A Hausdorff topological space (X, τ) is completely regular if and only if* $\tau = \mathrm{w}(X, C(X, \mathbb{R})) = \mathrm{w}(X, C_b(X, \mathbb{R}))$.

Proof. $\Longrightarrow$: Let $U \in \tau$ and $x \in U$. Since X is completely regular, we can find $f \in C(X, \mathbb{R})$ such that $f(x) = 0$ and $f\big|_{X\setminus U} = 1$. Let $V = \{u \in X : f(u) < 1\}$. Then V is $\mathrm{w}(X, C(X, \mathbb{R}))$-

open, $V \subseteq U$, and $x \in V$. Therefore, U is $\mathrm{w}(X, C(X, \mathbb{R}))$-open and so we infer that

$$\tau \subseteq \mathrm{w}(X, C(X, \mathbb{R})) . \tag{1.3.5}$$

From Definition 1.3.1 it is clear that we always have $\mathrm{w}(X, C(X, \mathbb{R})) \subseteq \tau$. This along with (1.3.5) and Proposition 1.3.8 yields $\tau = \mathrm{w}(X, C(X, \mathbb{R})) = \mathrm{w}(X, C_{\mathrm{b}}(X, \mathbb{R}))$.

$\Longleftarrow$: Let $C \subseteq X$ be closed and $x \notin C$. Then $U = X \setminus C \in \mathcal{N}_{\mathrm{w}}(x)$ where $\mathcal{N}_{\mathrm{w}}(x)$ is the family of weak neighborhoods of x. So we can find $V = \bigcap_{i=1}^{n}\{u \in X\colon |f_i(u) - f_i(x)| < 1\}$, $f_i \in C(X, \mathbb{R})$ for all $i \in \{1, \ldots, n\}$, such that $x \in V \subseteq U$. For each $i \in \{1, \ldots, n\}$ we define $g_i(u) = \min\{1, |f_i(u) - f_i(x)|\}$ and set $g = \max_{1 \le i \le n} g_i$. Obviously $g\colon X \to [0, 1]$ is continuous and $g(x) = 0$ as well as $g|_C = 1$. This proves that X is completely regular. $\square$

A weak topology of special interest is the product topology. So, let $\{(X_i, \tau_i)\}_{i \in I}$ be a family of Hausdorff topological spaces. Let $X = \prod_{i \in I} X_i$. The generic element $x \in X$ is denoted by $x = (x_i)$. For every $i \in I$ let $p_i\colon X \to X_i$ be defined by $p_i(x) = x_i$ where p_i is the projection map in the $i^{\underline{\text{th}}}$-component of the Cartesian product.

Definition 1.3.10. The **product topology** on X is the weak topology $\mathrm{w}(X, \{p_i\})$.

Remark 1.3.11. A basic element for the product topology has the form $V = \prod_{i \in I} V_i$ with $V_i \in \tau_i$ for all $i \in I$ and $V_i = X_i$ for all but a finite number of i's. In addition, note that $x^\alpha = (x_i^\alpha) \to x = (x_i)$ in $X = \prod_{i \in I} X_i$ if and only if $x_i^\alpha \to x_i$ for all $i \in I$. Note that if $A_i \subseteq X_i$ then $\overline{\prod_{i \in I} A_i} = \prod_{i \in I} \overline{A}_i$ and each projection map p_i is open.

Proposition 1.3.12. *$X = \prod_{i \in I} X_i$ with the product topology is Hausdorff.*

Proof. Recall that each X_i is Hausdorff. Let $x = (x_i) \in X$ and $u = (u_i) \in X$ with $x \neq u$. Then we can find at least one $i_0 \in I$ such that $x_{i_0} \neq u_{i_0}$. We can find $U_{i_0}, V_{i_0} \in \tau_{i_0}$ such that $x_{i_0} \in U_{i_0}$, $u_{i_0} \in V_{i_0}$ and $U_{i_0} \cap V_{i_0} = \emptyset$. Let $U = p_{i_0}^{-1}(U_{i_0})$ and $V = p_{i_0}^{-1}(V_{i_0})$. Then both are open in the product topology and $x \in U$, $u \in V$ and $U \cap V = \emptyset$. This implies that X is Hausdorff with the product topology. $\square$

Proposition 1.3.13. *If $\{(X_i, \tau_i)\}_{i \in I}$ is a family of Hausdorff topological spaces, then $X = \prod_{i \in I} X_i$ endowed with the product topology is regular if and only if (X_i, τ_i) is regular for each $i \in I$.*

Proof. $\Longrightarrow$: Each X_i is homeomorphic to a slice of $X = \prod_{i \in I} X_i$. Hence, the implication follows from Proposition 1.2.10.

$\Longleftarrow$: Let $x = (x_i) \in X = \prod_{i \in I} X_i$ and let U be any subbasic neighborhood of x. Then $U = \prod_{i \in I} V_i$ with $V_i = X_i$ for all $i \in I \setminus \{i_0\}$, $V_{i_0} \in \tau_{i_0}$. Exploiting the regularity of X_{i_0} we can find $W_{i_0} \in \tau_{i_0}$ such that

$$x_{i_0} \in W_{i_0} \subseteq \overline{W}_{i_0} \subseteq V_{i_0} , \tag{1.3.6}$$

see Proposition 1.2.8. Let $W = \prod_{i \in I} W_i$ with $W_i = X_i$ for all $i \in I \setminus \{i_0\}$ and W_{i_0} as above. Then W is open in the product topology and because of Remark 1.3.11 as well as (1.3.6), it follows that

$$x \in W \subseteq \overline{W} = \prod_{i\in I} \overline{W}_i \subseteq \prod_{i\in I} V_i = V\,.$$

This proves that $X = \prod_{i\in I} X_i$ is regular with the product topology; see Proposition 1.2.8. ☐

The Cartesian product of normal spaces need not be normal. For a counterexample, see Dugundji [91, p. 145]. However, we have the following result.

Proposition 1.3.14. *If $\{(X_i, \tau_i)\}_{i\in I}$ is a family of Hausdorff topological spaces and $X = \prod_{i\in I} X_i$ endowed with the product topology is normal, then (X_i, τ_i) is normal for each $i \in I$.*

Proof. Note that for each $i \in I$, X_i is homeomorphic to a slice of $X = \prod_{i\in I} X_i$, which is closed, and hence normal due to Proposition 1.2.15(a). Then the result follows from Proposition 1.2.15(b). ☐

Next we will consider the complementary situation to the one that led to the weak topology. So, let X, Y be topological spaces and $f\colon X \to Y$ be a continuous map. If we weaken the topology on Y we preserve the continuity of f. Hence, we want to identify the largest topology on Y for which f remains continuous.

Definition 1.3.15. Let (X, τ) be a topological space, Y a set, and $f\colon X \to Y$ a surjection. The **quotient topology** on Y induced by f is $\tau_q = \{U \subseteq Y : f^{-1}(U) \in \tau\}$. When Y is endowed with the quotient topology, then we say that f is a **quotient map**.

Remark 1.3.16. The quotient topology on Y makes f continuous and it is clearly the largest topology on Y that does this.

Proposition 1.3.17. *If (X, τ_X), (Y, τ_Y) are topological spaces and $f\colon X \to Y$ is supposed to be a continuous, open surjection, then f is a quotient map, that is $\tau_Y = \tau_q$.*

Proof. By definition $\tau_Y \subseteq \tau_q$. On the other hand, if $U \in \tau_q$, then $f^{-1}(U) \in \tau_X$ and since f is open, we have $U = f(f^{-1}(U)) \in \tau_Y$ and so $\tau_q \subseteq \tau_Y$. Therefore $\tau_Y = \tau_q$. ☐

Corollary 1.3.18. *If $\{(X_i, \tau_i)\}_{i\in I}$ are Hausdorff topological spaces and $X = \prod_{i\in I} X_i$ is endowed with the product topology, then $\tau_i = \tau_q$ for each $i \in I$.*

Proof. Just recall that each projection map $p_i\colon X = \prod_{i\in I} X_i \to X_i$ is a continuous open surjection. ☐

Proposition 1.3.19. *If (X, τ_X), (Y, τ_Y) are topological spaces and $f\colon X \to Y$ is supposed to be a continuous, closed surjection, then f is a quotient map, that is $\tau_Y = \tau_q$.*

Proof. Recall that $\tau_Y \subseteq \tau_q$. Let $U \in \tau_q$. Then $f^{-1}(U) \in \tau_X$ and so $X \setminus f^{-1}(U) =: C \subseteq X$ is closed. Since f is closed, we have that $f(C) \subseteq Y$ is τ_Y-closed. Note that $U = Y \setminus f(C) \in \tau_Y$. Hence $\tau_q \subseteq \tau_Y$ and we conclude that $\tau_Y = \tau_q$. ☐

The next proposition gives a criterion to recognize when a function defined on a quotient space is continuous.

Proposition 1.3.20. *If (X, τ_X), (Y, τ_Y), and (Z, τ_Z) are topological spaces, $f: X \to Y$ is a quotient map and $g: Y \to Z$, then g is continuous if and only if $g \circ f$ is continuous.*

Proof. $\Longrightarrow$: This follows from Proposition 1.1.36(a).

$\Longleftarrow$: Let $U \in \tau_Z$. Then $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \in \tau_X$. Hence $g^{-1}(U) \in \tau_Y$ since f is a quotient map, see Definition 1.3.15. This proves the continuity of g. □

Now we will show that the whole topic of the quotient topology can be covered by considering Y to be X/R with R being an equivalence relation; see Definition 1.2.30(b). Suppose $f: X \to Y$ is a surjection and define the relation $R \subseteq X \times X$ by setting xRx' if and only if $f(x) = f(x')$.

Let $e(x)$ be the equivalence class for x. Evidently $f|_{e(x)}$ is constant. Then the map $\hat{f}: X/R \to Y$ defined by $\hat{f}(e(x)) = f(x)$ is actually well-defined and a bijection. Note that if $e(x) = e(x')$, then $f(x) = f(x')$. In order to topologize X/R consider the standard quotient map $e: X \to X/R$ and consider the quotient topology induced by e. Then we have the following result.

Proposition 1.3.21. *If X is a topological space, Y is a set, $f: X \to Y$ is a surjection and R is the equivalence relation defined above, then X/R and Y are homeomorphic when both are endowed with the quotient topology.*

Remark 1.3.22. Instead of using the equivalence relation we may assume that X is partitioned by a collection $\mathcal{C}$ of disjoint subsets. Then we define an equivalence relation by setting xRu if and only if x, u are in the same element of $\mathcal{C}$. Then we can consider X/R. The simplest kind of quotient space can be obtained by the equivalence relation R in which only one equivalence class has more than one element $e(x_0) = A$ and for all other equivalence classes we have $e(x) = \{x\}$ with $x \in X \setminus A$. Then X/R is denoted by X/A and we obtain the quotient (identification) space by collapsing A to a single element $\{x_0\}$.

Example 1.3.23. (a) The quotient space of $[0, 1]$ obtained by identifying 0 and 1 is homeomorphic to a circle.

(b) The quotient space of $I^2 = [0, 1] \times [0, 1]$ by identifying the boundary with a single point is homeomorphic to a sphere in $\mathbb{R}^3$.

(c) The quotient space of $I^2 = [0, 1] \times [0, 1]$ by identifying the points $(0, x_2)$ and $(1, 1 - x_2)$ with $0 \le x_2 \le 1$ is homeomorphic to the **Möbius strip**.

(d) Let $X = I^2 = [0, 1] \times [0, 1]$ and consider an equivalence relation $R \subseteq X \times X$ defined as follows:

$$(x_1, 0)R(x_1, 1) \quad \text{for every } 0 \le x_1 \le 1\,, \tag{1.3.7}$$

$$(0, x_2)R(1, x_2) \quad \text{for every } 0 \le x_2 \le 1\,. \tag{1.3.8}$$

Then the quotient space is realized in two steps and gives a space homeomorphic to the torus. The first step is determined by (1.3.7), which produces a cylinder and then in the second step determined by (1.3.8), where we identify the two bases of the cylinder to generate the torus.

(e) If we replace (1.3.8) in the example above by

$$(0, x_2)R(1, 1 - x_2) \quad \text{for every } 0 \leq x_2 \leq 1\,,$$

then the resulting quotient space X/R is the **Klein bottle** .

(f) Let $D \subseteq \mathbb{R}^2$ be the unit disc, that is, $D = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1^2 + x_2^2 \leq 1\}$ and consider the equivalence relation

$$xR(-x) \quad \text{for all } \partial D = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1^2 + x_2^2 = 1\}\,.$$

That means, diametrically opposite points are identified. Then the quotient space D/R is called the **projective plane** and is denoted by P^2. One can proceed similarly to define P^n for any $n \in \mathbb{N}_0$ as the space obtained from $S^n = \{x \in \mathbb{R}^{n+1} \colon |x| = 1\}$ by identifying each point x with its antipode $-x$. The space P^n is known as the **projective n-space**.

1.4 Connectedness and Compactness

The property of connectedness says that the space has only one piece. It is a very important topological invariant with important applications in many other branches of mathematics. It is not difficult to come up with a definition of this very intuitive notion.

Definition 1.4.1. Let X be a topological space. A **separation** of X is a pair (U, V) of disjoint, nonempty, open sets of X such that $X = U \cup V$. If such a separation exists, we say that the space is **disconnected**. If there is no such separation for X, then we say that the space is **connected**. A set $A \subseteq X$ is connected, if it is a connected space when endowed with the subspace topology. Note that in a separation the two sets are both open and closed. We say that they are **clopen**.

Example 1.4.2. (a) The space $(\mathbb{R}, \tau_u)$, see Remark 1.2.23, is disconnected and the sets

$$\{x \in \mathbb{R} \colon x > \lambda\} \quad \text{and} \quad \{x \in \mathbb{R} \colon x \leq \lambda\}$$

with $\lambda \in \mathbb{R}$ form a separation of $\mathbb{R}$.

(b) The rationals $\mathbb{Q}$ with the relative Euclidean topology form a disconnected space. The sets

$$\{x \in \mathbb{R} \colon x > \pi\} \cap \mathbb{Q} \quad \text{and} \quad \{x \in \mathbb{R} \colon x < \pi\} \cap \mathbb{Q}$$

form a separation of $\mathbb{Q}$.

(c) A discrete space that is not a singleton is disconnected and the empty set is disconnected since there are no open sets to form a separation of it.

(d) $\mathbb{R} \setminus \{0\}$ is disconnected since $(-\infty, 0)$ and $(0, +\infty)$ form a separation. Similarly, $\mathbb{R}^2 \setminus \mathbb{R}$ is disconnected and we can have a separation using the sets

$$U = \{(x_1, x_2) \in \mathbb{R} \colon x_2 > 0\} \quad \text{and} \quad V = \{(x_1, x_2) \in \mathbb{R} \colon x_2 < 0\}\,.$$

Here, U is called the **upper half plane** and V is said to be the **lower half plane**.

(e) $\mathbb{R}$, endowed with the Euclidean topology, is connected. To show this we argue by contradiction. So, suppose that $\mathbb{R}$ is disconnected and (U, V) is a separation of $\mathbb{R}$. Let $x \in U$ and $y \in V$ and assume without loss of generality that $x < y$. Then $\hat{U} = U \cap [x, y]$ is closed and bounded in $\mathbb{R}$. Hence, $\hat{u} = \sup \hat{U} \in \hat{U}$. Furthermore, $\hat{u} \notin V$ since U and V are disjoint. Therefore $\hat{u} < y$ and $(\hat{u}, y] \subseteq V$. Thus, $\hat{u} \in \overline{V}$ and so $\hat{u} \in V$. It follows that $\hat{u} \in U \cap V$, a contradiction. This proves the connectedness of $\mathbb{R}$.

Remark 1.4.3. From the examples 1.4.2(b) and (e) we see that connectedness is not a hereditary property.

Proposition 1.4.4. *The connected subsets of* $\mathbb{R}$ *are singletons and intervals (open, closed, or half-open).*

Proof. Clearly singletons are connected. In addition, the argument in Example 1.4.2(e) shows that intervals are connected. It remains to show if $A \subseteq \mathbb{R}$ is connected, then A is an interval. If A is not an interval, then we can find $x, y \in A$ and $u \notin A$ such that $x < u < y$. Then $U = A \cap \{v \in \mathbb{R} \colon v < c\}$ and $V = A \cap \{v \in \mathbb{R} \colon v > c\}$ are a separation of A, a contradiction. □

Proposition 1.4.5. *Let X be a topological space. The following statements are equivalent:*

(a) *X is disconnected.*

(b) *There is a nonempty, proper subset of X, which is both open and closed.*

(c) *There is a continuous function from X into the two-point space $\{a, b\}$.*

(d) *X has a nonempty, proper subset A such that $\overline{A} \cap \overline{(X \setminus A)} = \emptyset$.*

Proof. (a) $\Longrightarrow$ (b): Since X is disconnected, it admits a separation (U, V). Then U as well as V are nonempty clopen.

(b) $\Longrightarrow$ (a): Suppose A is a proper, nonempty subset of X that is clopen. Let $C = X \setminus A$. Then (A, C) is a separation of X and so X is disconnected.

(a) $\Longrightarrow$ (c): Let (U, V) be a separation of X. Then the function $f \colon X \to \{a, b\}$ defined by

$$f(x) = \begin{cases} a & \text{if } x \in U\,, \\ b & \text{if } x \in V \end{cases}$$

is continuous.

(c) $\Longrightarrow$ (a): Since $f \colon X \to \{a, b\}$ is continuous, then $U = f^{-1}(a)$ and $V = f^{-1}(b)$ are disjoint, open sets in X such that $X = U \cup V$. So, (U, V) is a separation of X and we conclude that X is disconnected.

(a) $\Longrightarrow$ (d): Let (U, V) be a separation of X. Then $\overline{U} \cap \overline{V} = U \cap V = \emptyset$. So $\overline{U} \cap \overline{(X \setminus U)} = \emptyset$.

(d) $\Longrightarrow$ (a): We have that $\overline{A}$ and $\overline{(X \setminus A)}$ are disjoint, closed sets whose union is X. Hence $\overline{A}$ and $\overline{(X \setminus A)}$ are also open and form a separation of X. □

Corollary 1.4.6. *Let X be a topological space. The following statements are equivalent:*
(a) *X is connected.*
(b) *The only subsets of X that are open and closed are $\emptyset$ and X.*
(c) *There is no continuous function from X onto the two-point space $\{a, b\}$.*
(d) *X has no nonempty, proper subset A such that $\overline{A} \cap \overline{(X \setminus A)} = \emptyset$.*

Proposition 1.4.7. *If X, Y are topological spaces, X is connected and $f\colon X \to Y$ is continuous, then $f(X)$ is connected.*

Proof. Since $f\colon X \to f(X)$ is continuous we may assume that f is a continuous surjection. Arguing by contradiction, suppose that $Y = f(X)$ is disconnected and let (U, V) be a separation of Y. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint, open sets in X such that $X = f^{-1}(U) \cup f^{-1}(V)$. Hence X is disconnected, a contradiction. □

Remark 1.4.8. The last proposition gives at once that all open intervals in $\mathbb{R}$ are connected. Indeed recall that every open interval is homeomorphic to $\mathbb{R}$ and that $\mathbb{R}$ is connected; see Example 1.4.2(e).

If on a connected set A we adjoin some of its limit points we preserve connectedness.

Proposition 1.4.9. *If X is a topological space, $A \subseteq X$ is connected and $A \subseteq C \subseteq \overline{A}$, then C is connected.*

Proof. Arguing by contradiction, suppose that C is disconnected. Hence by Proposition 1.4.5, there exists a continuous surjection $f\colon C \to \{0, 1\}$. Since A is connected from Corollary 1.4.6 we have that $f(A) = \{0\}$ or $f(A) = \{1\}$. To fix things assume that $f(A) = \{0\}$. From Proposition 1.1.35 we have $f(\overline{A}) \subseteq \overline{f(A)} = \{0\}$. Hence, $f(C) = \{0\}$, a contradiction. □

Corollary 1.4.10. *If X is a topological space and $A \subseteq X$ is connected, then $\overline{A}$ is connected as well.*

Another useful result in determining whether or not a given subset is connected, is the following one.

Proposition 1.4.11. *If (X, τ) is a topological space and $A \subseteq X$, then A is disconnected if and only if there exist open sets $U, V \in \tau$ such that*

$$U \cap A \neq \emptyset\,, \quad V \cap A \neq \emptyset\,, \quad U \cap V \cap A = \emptyset\,, \quad \textit{and} \quad A \subseteq U \cup V\,.$$

Proof. $\Longrightarrow$: We have $A = \hat{U} \cup \hat{V}$ with $\hat{U}, \hat{V} \in \tau(A)$ with the subspace topology $\tau(A)$ and

$$\hat{U} = U \cap A \quad \text{as well as} \quad \hat{V} = V \cap A \quad \text{with} \quad U, v \in \tau\,.$$

Then we can easily check that U and V have the desired properties.

$\Longleftarrow$: Let $\hat{U} = U \cap A \neq \emptyset$ and $\hat{V} = V \cap A \neq \emptyset$. We have that $\hat{U}, \hat{V} \in \tau(A)$ and they are disjoint with $A = \hat{U} \cup \hat{V}$. Therefore, A is disconnected. □

It is obvious that connectedness is not preserved by arbitrary unions. Additional restrictions are needed.

Proposition 1.4.12. *If X is a topological space and $\{A_i\}_{i\in I}$ is any family of connected subsets of X such that $\bigcap_{i\in I} A_i \neq \emptyset$, then $\bigcup_{i\in I} A_i$ is connected.*

Proof. Let $C = \bigcup_{i\in I} A_i$. Suppose that C is disconnected. Then by Proposition 1.4.5 we can find a continuous map $f\colon C \to \{0, 1\}$. Since each A_i is connected, $f|_{A_i}$ is not surjective for all $i \in I$. Let $x_0 \in \bigcap_{i\in I} A_i$. Then $f(x) = f(x_0)$ for all $x \in A_i$ and for all $i \in I$. So, f is not surjective, a contradiction. □

Connectedness is preserved by arbitrary Cartesian products.

Proposition 1.4.13. *If $\{X_i\}_{i\in I}$ is an arbitrary family of nonempty, connected topological spaces, then $X = \prod_{i\in I} X_i$, endowed with the product topology, is connected as well.*

Proof. Arguing by contradiction, suppose that X is disconnected. So there is a continuous map $f\colon X \to \{0, 1\}$. Fix $u = (u_i)_{i\in I} \in X$ and let $i_1 \in I$. We define $f_{i_1}: X_{i_1} \to X$ by setting $f_{i_1}(x_i) = y = (y_i)_{i\in I}$ with $y_i = u_i$ for $i \neq i_1$ and $y_{i_1} = x_{i_1}$. Evidently f_{i_1} is continuous, which implies the continuity of $f \circ f_{i_1}: X_{i_1} \to \{0, 1\}$. By hypothesis, X_{i_1} is connected. So, $f \circ f_{i_1}$ is constant and $(f \circ f_{i_1})(x_{i_1}) = f(u)$ for every $x_{i_1} \in X_{i_1}$. Hence $f(x) = f(u)$ for all $x \in X$, which are equal to u except for the i_1-component. We repeat this process with another index $i_2 \in I$. Continuing this way we see that $f(x) = f(u)$ for all $x \in X$, which are equal to u except on a finite number of coordinates. This set is dense in X and so by Proposition 1.1.35(b), f is constant, a contradiction. This proves that X is connected. □

Corollary 1.4.14. *The space $\mathbb{R}^n$ with $n \in \mathbb{N}$ is connected.*

Example 1.4.15. Let $A = \{(0, y) \in \mathbb{R}^2 : 0 \le y \le 1\}$ and $C = \{(x, y) \in \mathbb{R}^2 : 0 < x \le 1, y = \sin \pi/x\}$. Evidently C is connected because of Propositions 1.4.7 and 1.4.13. Furthermore, $S = \overline{C} = A \cup C$ is connected; see Corollary 1.4.10. The set S is known as the **topologist's sine curve**.

Remark 1.4.16. It is clear that intersection of even two connected spaces need not be connected. Furthermore, suppose that $\{A_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of connected spaces. Then $\bigcap_{n\ge1} A_n$ need not be connected. To see this, let $X = I^2 \setminus \{(x, 0) : 1/2 \le x \le 2/3\}$ with $I = [0, 1]$ and $A_n = \{(x, y) \in X : y \le 1/n\}$ with $n \in \mathbb{N}$.

A disconnected space can be decomposed in a unique way into connected **components** and the number of components can be viewed as an indication of how disconnected the space is.

Definition 1.4.17. A **component** of a topological space X is a maximal connected subset C of X. That is, C is connected and it is not properly contained in a connected subset of X.

Remark 1.4.18. A component is necessarily closed. Indeed, from Corollary 1.4.10 we know that $\overline{C}$ is connected. The maximality of C implies that $C = \overline{C}$. Hence, C is closed. The family of distinct components of X form a partition of X. To see this, note if C, C' are two distinct components of X and $C \cap C' \neq \emptyset$, then from Proposition 1.4.12 we have that $C \cup C'$ is connected, contradicting the maximality of the components. Moreover, for the same reason, each $x \in X$ belongs in a unique component. Given $x \in X$ let $C(x)$ denote the component of X containing x. Then, for points $x, u \in X$, $C(x)$ and $C(u)$ are either identical or disjoint. Every connected subset of X is contained in one component and X is connected if and only if it has only one component. Finally if $\{U, V\}$ is a separation of X and C is a component of X, then $C \subseteq U$ or $C \subseteq V$.

Taking into account the remarks above and Proposition 1.4.19, we infer the following result.

Proposition 1.4.19. *If X, Y are topological spaces and $f\colon X \to Y$ is continuous, then the image of each component of X lies in a component of Y.*

Remark 1.4.20. In particular, a homeomorphism f induces a 1-1 correspondence between the components of X and Y with $C(x)$ being homeomorphic to $C(f(x))$ for all $x \in X$.

Definition 1.4.21. (a) A topological space X is **totally disconnected** provided that each component of X is a singleton.

(b) A point $x \in X$ is a **cut point** of a connected topological space X provided that $X \setminus \{x\}$ is disconnected. We say that $x \in X$ is an n**-cut point** provided that $X \setminus \{x\}$ has n-components.

From Proposition 1.4.19 it follows the following result.

Proposition 1.4.22. *Homeomorphic spaces have the same number of cut points of each type.*

From an analytical point of view, the notion of path-connectedness is more natural. Path-connectedness is a topological property stronger than connectedness and it is useful in many applications. It is a very intuitive notion that in a path-connected space any two distinct points can be joined by a continuous path in the space.

Definition 1.4.23. (a) A **path** in a topological space X is a continuous map $\sigma\colon [0, 1] \to X$. We say that $\sigma(0)$ is the **initial point** of the path and $\sigma(1)$ is the **final point** of the path. The set $\sigma([0, 1]) \subseteq X$ is called a **curve** in X. If σ is a path in X, then $\overline{\sigma}(t) = \sigma(1 - t)$ for all $t \in [0, 1]$ is the **reverse path**.

(b) A topological space X is said to be **path-connected** provided that for each pair of points $x, u \in X$ there is a path in X with initial point x and final point u. A subset C of X is **path-connected** if C has this property for the subspace topology.

The next proposition compares connectedness and path-connectedness.

Proposition 1.4.24. *Every path-connected topological space is connected.*

Proof. Suppose that X is path-connected and let $u \in X$. For each $x \in X$, let σ_x be the path in X with initial point u and final point x. Let $C_x = \sigma_x([0, 1])$ be the corresponding curve. From Proposition 1.4.7 we know that $C_x \subseteq X$ is connected. Note that $u \in \bigcap_{x \in X} C_x$. So, from Proposition 1.4.12, it follows that $\bigcup_{x \in X} C_x = X$ is connected. ☐

Remark 1.4.25. The converse of the above is not true in general. As a counterexample, consider the topologist's sine curve $S = A \cup C$ introduced in Example 1.4.15. Then S is connected but not path-connected. To prove that S is not path-connected, we show that it is not possible to join a point in A to a point in C by a path in S. To this end, let $a \in A$ and let $\sigma\colon [0, 1] \to X$ be a path with initial point a. Note that A is closed in S (see Proposition 1.1.27), and so $\sigma^{-1}(A) \subseteq [0, 1]$ is closed and nonempty, since $0 \in \sigma^{-1}(A)$. Let $t \in \sigma^{-1}(A)$ and choose a small $\varepsilon > 0$ such that $\sigma((t - \varepsilon, t + \varepsilon)) \subseteq \overline{B}_{1/2}(\sigma(t)) = \{u \in \mathbb{R}^2 : |u - \sigma(t)| \leq 1/2\}$, which is possible since σ is continuous. Note that $S \cap \overline{B}_{1/2}(\sigma(t))$ consists of a closed interval on the y-axis of $\mathbb{R}^2$ together with parts of the curve $y = \sin(\pi/x)$, each of which is homeomorphic to a closed interval. Moreover, any two of these parts are disjoint in $S \cap \overline{B}_{1/2}(\sigma(t))$. So $A \cap \overline{B}_{1/2}(\sigma(t))$ is a component of $S \cap \overline{B}_{1/2}(\sigma(t))$. Since $\sigma(t) \in A \cap \overline{B}_{1/2}(\sigma(t))$ and $(t - \varepsilon, t + \varepsilon)$ is connected, we must have $\sigma((t - \varepsilon, t + \varepsilon)) \subseteq A \cap \overline{B}_{1/2}(\sigma(t))$. This shows that $\sigma^{-1}(A) \subseteq [0, 1]$ is open. Hence $\sigma^{-1}(A) = [0, 1]$ being both closed and open. So, $\sigma([0, 1]) \subseteq A$ and this proves that S cannot be path-connected.

Proposition 1.4.26. *If X is a topological space and $u \in X$, then X is path-connected if and only if each $x \in X$ can be joined to u by a path.*

Proof. $\Longrightarrow$: This is obvious.

$\Longleftarrow$: Let $x, x' \in X$ and consider the paths $\sigma, \sigma'\colon [0, 1] \to X$ such that σ has initial point x and final point u as well as σ' having initial point u and final point x'. We define $\hat{\sigma}\colon [0, 1] \to X$ by

$$\hat{\sigma}(t) = \begin{cases} \sigma(2t) & \text{if } t \in [0, \frac{1}{2}]\,, \\ \sigma'(2t - 1) & \text{if } t \in [\frac{1}{2}, 1]\,. \end{cases}$$

This is a continuous path since $\sigma(1) = \sigma'(0) = u$; see Proposition 1.1.37. Moreover, $\hat{\sigma}(0) = x$ and $\hat{\sigma}(1) = x'$. Therefore, X is path-connected. ☐

Definition 1.4.27. Let σ_1, σ_2 be two paths in X such that $\sigma_1(t) = \sigma_2(0)$. The **path composition** of σ_1 and σ_2 denoted by $\sigma_1 * \sigma_2$ is the path in X defined by

$$(\sigma_1 * \sigma_2)(t) = \begin{cases} \sigma_1(2t) & \text{if } t \in [0, \frac{1}{2}]\,, \\ \sigma_2(2t - 1) & \text{if } t \in [\frac{1}{2}, 1]\,. \end{cases}$$

The next result is a straightforward consequence of Definition 1.4.23(b) and of Proposition 1.1.36(a).

Proposition 1.4.28. *If X, Y are topological spaces, X is path-connected, and $f : X \to Y$ is continuous, then $f(X)$ is path-connected.*

Remark 1.4.29. It follows that path-connectedness is a topological invariant. In contrast to connectedness, see Corollary 1.4.10, the closure of a path-connected set need not be path-connected. We consider the topologist's sine curve from Example 1.4.15. We have $S = \overline{C}$ and C is path-connected; see Proposition 1.4.28. However, we proved that S is not path-connected; see Remark 1.4.25.

Many results about connectedness have analogues for path connectedness.

Proposition 1.4.30. *If X is a topological space and $\{A_i\}_{i\in I}$ is any family of path-connected subsets of X such that $\bigcap_{i\in I} A_i \neq \emptyset$, then $\bigcup_{i\in I} A_i$ is path-connected.*

Proof. Let $x \in \bigcup_{i\in I} A_i$ and pick $u \in \bigcap_{i\in I} A_i$. Since $x \in A_{i_0}$ for some $i_0 \in I$, we can join x and u by a path in X since A_{i_0} is path-connected. Proposition 1.4.26 implies that $\bigcup_{i\in I} A_i$ is path-connected. □

Proposition 1.4.31. *If $\{X_i\}_{i\in I}$ is an arbitrary family of nonempty, path-connected topological spaces, then $X = \prod_{i\in I} X_i$ endowed with the product topology is path-connected as well.*

Proof. Let $x = (x_i)$, $u = (u_i) \in X$. For each $i \in I$, X_i is path-connected so we can find a path σ_i with initial point x_i and final point u_i. Then $\sigma = (\sigma_i)$ is a path in X joining x and u; see Proposition 1.3.4. Hence, X is path-connected as well. □

Definition 1.4.32. A **path component** of a topological space is a maximal path-connected subset C of X. That is, C is path-connected and it is not properly contained in a path-connected subset of X.

Remark 1.4.33. Path components have almost the same properties as components. So every $x \in X$ belongs to exactly one path component denoted by $P(x)$. If $x \neq x'$, then $P(x) \cap P(x') = \emptyset$ or $P(x) = P(x')$. Every path-connected set $C \subseteq X$ is contained in a path component and X is path-connected if and only if X has only one path component. Note that we said almost the same properties. The reason for this, in contrast to components, is that path components need not be closed. Consider the topologist's sine curve $S = A \cup C$, see Example 1.4.15. Then A and C are the path components of S but C is not closed; recall that $\overline{C} = S$. A path component of X is a subset of some component of X.

Connectedness and path-connectedness are global topological properties since they concern the whole topological space. Local topological properties concern the structure of the space near a particular point, if we recall the notion of first countability; see Definition 1.2.20(a). In the next definition we provide local versions of the notions of connectedness and of path-connectedness.

Definition 1.4.34. A topological space X is said to be **locally connected** (resp. **locally path-connected**) if for every $x \in X$ and every $U \in \mathcal{N}(x)$ we can find a connected (resp. path-connected) $V \in \mathcal{N}(x)$ such that $V \subseteq U$.

Remark 1.4.35. Equivalently X is locally connected (resp. locally path-connected) if and only if every $x \in X$ has a local basis consisting of connected (resp. path-connected) sets. A space can be connected (resp. path-connected) without being locally connected (resp. locally path-connected). Consider the topologist's sine curve (see Example 1.4.15), which is connected but not locally connected. Of course local connectedness (resp. local path-connectedness) does not imply connectedness (resp. path-connectedness). Consider the union of two disjoint, closed balls in $\mathbb{R}^N$.

Proposition 1.4.36. *A topological space X is connected if and only if for each open set $U \subseteq X$ each component of U is open.*

Proof. $\Longrightarrow$: Let C be a component of the open set $U \subseteq X$. Given $x \in C$ we can find a connected open set $V_x \subseteq U$ with $x \in V_x$. We have $V_x \subseteq C$ and since $x \in C$ was arbitrary, we conclude that C is open.

$\Longleftarrow$: Let $x \in X$ and let $U \in \mathcal{N}(x)$. Then by hypothesis the component C of U containing x is open and so X is locally connected. □

Corollary 1.4.37. *If a topological space X is locally connected then every component of X is open (and closed).*

Proposition 1.4.38. *If X is a topological space, then the following statements are equivalent:*
(a) *Every path component of X is open, hence closed as well.*
(b) *Every point of X has a path-connected neighborhood.*

Proof. (a) $\Longrightarrow$ (b): Let $x \in X$ and let $C(x)$ be the path component containing x. By hypothesis $C(x)$ is open and so X is locally path-connected.

(b) $\Longrightarrow$ (a): Let C be a path component and $x \in C$. By hypothesis we can find a path-connected $U \in \mathcal{N}(x)$. Hence, $U \subseteq C$ and since $x \in C$ is arbitrary we conclude that C is open. Note that $X \setminus C$ is the union of the remaining open path components, as we just proved, and it is open, so C is closed. □

We saw that path-connectedness is stronger than connectedness; see Proposition 1.4.24. The next proposition provides conditions for the two notions to be equivalent.

Proposition 1.4.39. *A topological space X is path-connected if and only if X is connected and every $x \in X$ has a path-connected neighborhood.*

Proof. $\Longrightarrow$: This follows from Proposition 1.4.24 and the fact that X is a neighborhood of every $x \in X$, and by hypothesis it is path-connected.

$\Longrightarrow$: According to Proposition 1.4.38 every path component of X is open and closed in X. Since X is connected, it follows that it has only one path component, and hence X is path-connected. $\square$

Corollary 1.4.40. *An open subset of $\mathbb{R}^n$ is connected if and only if it is path-connected.*

Remark 1.4.41. The corollary above fails for nonopen sets in $\mathbb{R}^n$. To see this, consider the topologist's sine curve.

Now we pass to another fundamental topological notion, namely the notion of **compactness**. This concept is an abstraction to general topological spaces of a property of closed and bounded intervals, cf. the Heine–Borel Theorem. Compactness does not mean only small in size. It is more than that. For example the intervals $[0, 1]$ and $(0, 1)$ have the same size but $[0, 1]$ is compact while $(0, 1)$ is not. Compactness is important in analysis since it combines well with continuity.

Definition 1.4.42. Let X be a Hausdorff topological space. We say that X is **compact** if every open cover admits a finite subcover; see Definition 1.2.26. A subset $A \subseteq X$ is compact provided A, endowed with the relative subspace topology, is compact.

Remark 1.4.43. Since compact subsets of a non-Hausdorff space need not be closed (a rather awkward situation), we have included in the definition of compactness that X is Hausdorff. Since relatively open sets in A are of the form $U \cap A$ with $U \subseteq X$ open, the definition of compactness of $A \subseteq X$ takes the following form: "$A \subseteq X$ is compact if and only if every open cover of A by open sets in X admits a finite subcover."

Definition 1.4.44. Let X be a set and $\mathcal{L} \subseteq 2^X \setminus \{\emptyset\}$. We say that $\mathcal{L}$ has the **finite intersection property** if every finite subcollection of $\mathcal{L}$ has a nonempty intersection.

Proposition 1.4.45. *Let X be a Hausdorff topological space. The following statements are equivalent:*
(a) *X is compact.*
(b) *Every family of nonempty, closed subsets of X with the finite intersection property has a nonempty intersection.*
(c) *Every net in X has a convergent subnet in X.*

Proof. (a) $\Longrightarrow$ (b): Let $\mathcal{L}$ be a family of nonempty, closed subsets of X with the finite intersection property. If $\bigcap_{C\in\mathcal{L}} C = \emptyset$, then $X = \bigcup_{C\in\mathcal{L}}(X \setminus C)$ and so $\{X \setminus C\}_{C\in\mathcal{L}}$ is an open cover of X. The compactness of X implies that we can find a finite subcover such that $X = \bigcup_{k_1}^{n}(X \setminus C_k)$ with $n \in \mathbb{N}$. Then $\bigcap_{k=1}^{n} C_k = \emptyset$, contradicting the fact that $\mathcal{L}$ has the finite intersection property.

(b) $\Longrightarrow$ (a): Let $\mathcal{D}$ be an open cover of X. Then $X = \bigcup_{U\in\mathcal{D}} U$ and so $\bigcap_{U\in\mathcal{D}}(X \setminus U) = \emptyset$. This means that the finite intersection property does not hold for the collection $\{X \setminus U\}_{U\in\mathcal{D}}$ and so we can find $\{U_k\}_{k=1}^{n} \subseteq \mathcal{D}$ such that $\bigcap_{k=1}^{n}(X \setminus U_k) = \emptyset$. Hence, $X = \bigcup_{k=1}^{n} U_k$ and so we conclude that X is compact.

(b) $\implies$ (c): Let $\{x_i\}_{i\in I}$ be a net in X. Let $A_\alpha = \overline{\{x_i\}}_{i\geq\alpha}$ with $\alpha \in I$. Then $\{A_\alpha\}_{\alpha\in I}$ is a family of nonempty, closed subsets of X with the finite intersection property. So, by hypothesis we can find $x \in \bigcap_{\alpha\in I} A_\alpha$. Evidently, x is a cluster point of $\{x_i\}_{i\in I}$. So, using Proposition 1.2.36 we can find a subnet of $\{x_i\}_{i\in I}$ converging to $x \in X$.

(c) $\implies$ (b): Let $\mathcal{L}$ be a family of nonempty, closed subsets of X with the finite intersection property. Let $\mathcal{F}$ be the family of all finite intersections of members of $\mathcal{L}$. Then $\mathcal{F}$ has the finite intersection property and since $\mathcal{L} \subseteq \mathcal{F}$ it suffices to show that $\bigcap_{D\in\mathcal{F}} D \neq \emptyset$. Since the intersection of two elements in $\mathcal{F}$ is again an element of $\mathcal{F}$, we see that $\mathcal{F}$ is directed. Let $x_D \in D$ with $D \in \mathcal{F}$. Then $\{x_D\}_{D\in\mathcal{F}} \subseteq X$ is a net and so by hypothesis it has a cluster point x. Then $x \in D$ for all $D \in \mathcal{F}$ and so $\bigcap_{D\in\mathcal{F}} D \neq \emptyset$. ☐

Proposition 1.4.46. *If X is a compact topological space and $C \subseteq X$ is closed, then C is compact.*

Proof. Let $\mathcal{L}$ be a cover of C by sets open in X. Then $\mathcal{L}_0 = \mathcal{L} \cup (X \setminus C)$ is an open cover of X. Since X is compact, $\mathcal{L}_0$ has a finite subcover $\{U_k, X \setminus A\}_{k=1}^n$ with $U_k \in \mathcal{L}$. Then $C \subseteq \bigcup_{k=1}^n U_k$ and so C is closed; see Remark 1.4.43. ☐

Proposition 1.4.47. *If X is a Hausdorff topological space and $C \subseteq X$ is compact, then C is closed.*

Proof. Let $\{x_i\}_{i\in I} \subseteq C$ be a net such that $x_i \to x$. Since X is compact, we can find a subnet $\{u_\alpha\}_{\alpha\in I}$ such that $u_\alpha \to x \in C$; see Propositions 1.4.45 and 1.2.40. Therefore, we conclude that $C \subseteq X$ is compact. ☐

Corollary 1.4.48. *If X is a compact topological space and $A \subseteq X$, then A is compact if and only if A is closed.*

Proposition 1.4.49. *If X is Hausdorff topological space and K_1, K_2 are compact, disjoint subsets of X, then we can find open $U, V \subseteq X$ such that $K_1 \subseteq U$, $K_2 \subseteq V$ and $U \cap V = \emptyset$.*

Proof. First assume that $K_1 = \{u\}$ is a singleton. Then for each $x \in K_2$ we can find open sets $U_x, V_x \subseteq X$ such that $u \in U_x$, $x \in V_x$ and $U_x \cap V_x = \emptyset$ because X is Hausdorff. Then $\{V_x\}_{x\in K_2}$ is an open cover of K_2. The compactness of K_2 implies that we can find a finite subcover $\{V_{x_k}\}_{k=1}^n$. Let

$$U = \bigcap_{k=1}^n U_{x_k} \quad \text{and} \quad V = \bigcup_{k=1}^n V_{x_k} .$$

Both are open sets in X, $u \in U$ and $K_2 \subseteq V$. So, we have proven the proposition when K_1 is a singleton.

Now consider the case of a general compact set $K_1 \subseteq X$. From the previous part of the proof we know that for every $u \in K_1$ we can find open $U_u, V_u \subseteq X$ such that $u \in U_u$, $K_1 \subseteq V_u$ and $U_u \cap V_u = \emptyset$. Note that $\{U_u\}_{u\in K_1}$ is an open cover of K_1 and so by the compactness we can find a finite subcover $\{U_{u_k}\}_{k=1}^n$. Set

$$U = \bigcup_{k=1}^n U_{x_k} \quad \text{and} \quad V = \bigcap_{k=1}^n V_{x_k} \,.$$

Then both are open sets in X, $K_1 \subseteq U$, $K_2 \subseteq V$ and $U \cap V = \emptyset$. □

Corollary 1.4.50. *A compact topological space is normal.*

The next result is one of the main theorems on compactness.

Theorem 1.4.51. *If X, Y are Hausdorff topological spaces, $K \subseteq X$ is compact, and $f\colon X \to Y$ is continuous, then $f(K) \subseteq Y$ is compact.*

Proof. Let $\{V_i\}_{i\in I}$ be an open cover of $f(K)$. Then $\{f^{-1}(V_i)\}_{i\in I}$ is an open cover of K. The compactness of K implies the existence of a finite subcover $\{f^{-1}(V_{i_k})\}_{k=1}^n$, that is $K \subseteq \bigcup_{k=1}^n f^{-1}(V_{i_k})$. Hence

$$f(K) \subseteq f\left(\bigcup_{k=1}^n f^{-1}(V_{i_k})\right) = \bigcup_{k=1}^n f(f^{-1}(V_{i_k})) \subseteq \bigcup_{k=1}^n V_{i_k} \,.$$

Therefore, $f(K)$ is compact. □

In $\mathbb{R}$ the compact sets are closed and bounded; see the Heine–Borel Theorem. So, Theorem 1.4.51 yields the following result known as the “Weierstraß-Theorem.”

Theorem 1.4.52 (Weierstraß Theorem). *If X is a compact topological space and $f\colon X \to \mathbb{R}$ is continuous, then there exist $x_0, \hat{x} \in X$ such that*

$$f(x_0) = \inf[f(x)\colon x \in X] \quad \textit{and} \quad f(\hat{x}) = \sup[f(x)\colon x \in X] \,.$$

Remark 1.4.53. In addition, Theorem 1.4.51 implies that compactness is a topological property.

Theorem 1.4.54. *If X, Y are Hausdorff topological spaces, X is compact and $f\colon X \to Y$ is a continuous bijection, then f is a homeomorphism.*

Proof. Let $C \subseteq X$ be closed. Then C is compact because of Corollary 1.4.48. Taking into account Theorem 1.4.51, we conclude that $f(C) \subseteq Y$ is compact, hence closed as well; see Proposition 1.4.47. Therefore, f is a closed function and then by Proposition 1.1.42, f is a homeomorphism. □

Compactness is preserved by Cartesian products. This is the celebrated “Tychonoff's Product Theorem.” To prove this result, we need some preliminary material. First we present three statements of set theory that are equivalent.

Axiom of Choice: Let K be any set-valued map on a set X such that $K(x) \neq \emptyset$ for all $x \in X$. Then there is a function k on X such that $k(x) \in K(x)$ for all $x \in X$.

Zorn's Lemma: Let $(X, \leq)$ be a partially ordered set such that for every chain $C \subseteq X$ there is an upper bound $\hat{u} \in X$, that is, $x \leq \hat{u}$ for all $x \in C$. Then X has a maximal element, that is, there exists $x_0 \in X$ such that there is no $v \in X$ with $x_0 < v$; see Definition 1.2.30(c).

Hausdorff Maximal Principle: For every partially ordered set $(X, \leq)$ there is a maximal chain $C \subseteq X$.

Lemma 1.4.55. *If (X, τ) is a Hausdorff topological space and $\mathcal{L}_0$ is a collection of subsets of X with the finite intersection property, then there exists a maximal collection $\mathcal{L}$ of subsets of X with the finite intersection property and containing $\mathcal{L}_0$. Moreover, finite intersections of elements in $\mathcal{L}$ are again in $\mathcal{L}$ and every subset of X intersecting every set in $\mathcal{L}$ is in $\mathcal{L}$.*

Proof. The family of all collections of sets in X with the finite intersection property and containing $\mathcal{L}_0$ is partially ordered by inclusion. Therefore, the Hausdorff Maximal Principle implies the existence of a maximal chain $\mathcal{C}$. Let $\mathcal{L} = \bigcup_{a \in \mathcal{C}} a$.

Let $\{A_k\}_{k=1}^n \subseteq \mathcal{L}$. It belongs to at most n-collections a_k and $\{a_k\}_{k=1}^n$ is linearly ordered. So, there is a collection a_n that contains the others. Hence, $A_k \in a_n$ for all $k = 1, \ldots, n$ and $\bigcap_{k=1}^n A_k \neq \emptyset$ because of the finite intersection property. Thus, $\mathcal{L}$ has the finite intersection property. Note again that $\mathcal{L}$ is maximal.

Let $\mathcal{L}'$ be the collection of all finite intersections of sets in $\mathcal{L}$. Then $\mathcal{L}_0 \subseteq \mathcal{L}'$ and it has the finite intersection property. Hence, by maximality $\mathcal{L}' = \mathcal{L}$.

Finally, let $A \subseteq X$ be such that $A \cap D \neq \emptyset$ for all $D \in \mathcal{L}$. Then the collection $\mathcal{L}' = \mathcal{L} \cup \{A\}$ has the finite intersection property and contains $\mathcal{L}_0$. Therefore, by the maximality, $A \in \mathcal{L}$. □

We will use this lemma to prove "Tychonoff's Product Theorem."

Theorem 1.4.56 (Tychonoff's Product Theorem). *If $\{(X_i, \tau_i)\}_{i \in I}$ are compact topological spaces, then $X = \prod_{i \in I} X_i$ endowed with the product topology is compact.*

Proof. Let $\mathcal{L}_0$ be a collection of closed sets in X with the finite intersection property and let $\mathcal{L}$ be the maximal collection postulated by Lemma 1.4.55. Note that while the elements of $\mathcal{L}_0$ are closed, those of $\mathcal{L}$ need not be closed. We will show that

$$\bigcap_{D \in \mathcal{L}} \overline{D} \neq \emptyset\,. \tag{1.4.1}$$

For each $i \in I$, let $\mathcal{L}_i$ be the i-projection of $\mathcal{L}$, that is, $\mathcal{L}_i = \{p_i(D) \colon D \in \mathcal{L}\}$. The elements of this collection need not be open nor closed. However, since $\mathcal{L}$ has the finite intersection property, it follows that so does $\mathcal{L}_i$. Then $\overline{\mathcal{L}_i} = \{\overline{p_i(D)} \colon D \in \mathcal{L}\}$ has a nonempty intersection; see Proposition 1.4.45. Let $x_i \in \bigcap_{D \in \mathcal{L}} \overline{p_i(D)} \subseteq X_i$ and $x = (x_i) \in X$. We claim that $x \in \overline{D}$ for all $D \in \mathcal{L}$.

Let $U \in \mathcal{N}(x)$. Then from the definition of the product topology we know that we can find $i_1, \ldots, i_n \in I$, and $U_{i_k} \in \tau_{i_k}$ with $k = 1, \ldots, n$ such that

$$x \in \bigcap_{k=1}^{n} p_{i_k}^{-1}(U_{i_k}) \subseteq U .$$

Note that $x_{i_k} \in U_{i_k} \cap \overline{\mathcal{L}}_{i_k}$, hence $U_{i_k} \cap \mathcal{L}_{i_k} \neq \emptyset$. Therefore, $p_{i_k}^{-1}(U_{i_k}) \cap \mathcal{L} \neq \emptyset$. Thus, Lemma 1.4.55 implies that $p_{i_k}^{-1}(U_{i_k}) \in \mathcal{L}$. Hence $\bigcap_{k=1}^{n} p_{i_k}^{-1}(U_{i_k}) \in \mathcal{L}$. We conclude that (1.4.1) holds and this implies that X is compact; see Proposition 1.4.45. $\square$

Let us now introduce some generalizations of the notion of compactness.

Definition 1.4.57. Let (X, τ) be a Hausdorff topological space.
(a) We say that X is **countably compact** if every countable open cover has a finite subcover.
(b) We say that X is **limit point compact** (or that is has the **Bolzano–Weierstraß property**) if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ has at least one cluster point.
(c) We say that X is **sequentially compact** if every sequence has a τ-convergent subsequence.

Remark 1.4.58. Clearly, “Compactness” implies “Countable Compactness” and “Sequential Compactness” implies “Limit Point Compactness.” In general both implications are not reversible.

Combining Definition 1.4.57 and Proposition 1.4.45 gives the following result.

Proposition 1.4.59. *A Hausdorff topological space (X, τ) is countably compact if and only if every countable family of closed sets with the finite intersection property has a nonempty intersection.*

Proposition 1.4.60. *A Hausdorff topological space (X, τ) is countably compact if and only if it is limit point compact.*

Proof. $\Longrightarrow$: Let $\{x_n\}_{n\geq 1} \subseteq X$ and define $A_m = \overline{\{x_n\}_{n\geq m}}$ with $m \in \mathbb{N}$. Then $\{A_m\}_{m\geq 1}$ are closed sets with the finite intersection property. So, $\bigcap_{m\geq 1} A_m \neq \emptyset$ by Proposition 1.4.59. Any $x \in \bigcap_{m\geq 1} A_m \neq \emptyset$ is a cluster point of the sequence. Therefore, X is limit point compact.

$\Longleftarrow$: Let $\{C_n\}_{n\geq 1}$ be closed sets in X with the finite intersection property. Let $x_n \in \bigcap_{k=1}^{n} C_k$ with $n \in \mathbb{N}$. The limit point compactness of X implies that $\{x_n\}_{n\geq 1}$ has at least one cluster point x. Then $x \in \overline{\{x_n\}_{n\geq 1}} \subseteq \overline{\bigcap_{n\geq 1} C_n} = \bigcap_{n\geq 1} C_n \neq \emptyset$. Using Proposition 1.4.59, this implies that X is countably compact. $\square$

Corollary 1.4.61. *“Sequential Compactness” implies “Countable Compactness.”*

The reverse assertion is true under some additional assumptions.

Proposition 1.4.62. *If (X, τ) is a Hausdorff topological space that is first countable and countably compact, then X is sequentially compact.*

Proof. Let $\{x_n\}_{n\geq 1} \subseteq X$ and $x \in \overline{\{x_n\}_{n\geq 1}}$. Let $\{U_k\}_{k\in\mathbb{N}} \subseteq \mathcal{N}(x)$ such that $U_{k+1} \subseteq U_k$ for all $k \in \mathbb{N}$. Recall that X is first countable. Choose $x_m \in U_m \cap \{x_n\}_{n\geq 1}$ with $m \in \mathbb{N}$. Then $\{x_m\}_{m\geq 1}$ is a subsequence of $\{x_n\}_{n\geq 1}$ τ-converging to x. Therefore, X is sequentially compact. □

This proposition together with Lindelöf's Theorem (see Theorem 1.2.27), gives the following result.

Theorem 1.4.63. *If (X, τ) is a Hausdorff topological space that is second countable, then the following statements are equivalent:*
(a) *X is compact.*
(b) *X is countably compact.*
(c) *X is limit point compact.*
(d) *X is sequentially compact.*

Next we introduce a modification of compactness to a local property.

Definition 1.4.64. A Hausdorff topological space (X, τ) is said to be **locally compact** if for every $x \in X$ there exists $U \in \mathcal{N}(x)$ such that $\overline{U}$ is compact.

Remark 1.4.65. A set $A \subseteq X$ such that $\overline{A}$ is compact is said to be **relatively compact** (or **precompact**). The space $\mathbb{R}^N$ with the Euclidean topology is locally compact but not compact. Recall the Heine–Borel Theorem, which says that $A \subseteq \mathbb{R}^N$ is compact if and only if A is closed and bounded. Bounded means that there exists $r > 0$ such that $A \subseteq \overline{B}_r = \{u \in \mathbb{R}^N : |u| \leq r\}$.

Proposition 1.4.66. *Let (X, τ) be a Hausdorff topological space. The following statements are equivalent:*
(a) *X is locally compact.*
(b) *For every $x \in X$ and every $U \in \mathcal{N}(x)$ there is a relatively compact $V \in \mathcal{N}(x)$ such that $x \in V \subseteq \overline{V} \subseteq U$.*
(c) *For every compact K and $U \in \tau$ such that $U \supseteq K$, there exists a relatively compact $V \in \tau$ such that $K \subseteq V \subseteq \overline{V} \subseteq U$.*
(d) *X has a basis consisting of relatively compact open sets.*

Proof. (a) $\Longrightarrow$ (b): Let $x \in X$ and $U \in \mathcal{N}(x)$. Taking into account the local compactness of X we find $W \in \mathcal{N}(x)$ such that $\overline{W}$ is compact. Corollary 1.4.50 implies that $\overline{W}$ endowed with the relative topology is regular. Then $\overline{W} \cap U$ is a neighborhood of x in $\overline{W}$. Proposition 1.2.8 implies the existence of an open set $D \subseteq \overline{W}$ such that

$$x \in D \subseteq \overline{D}^{\overline{W}} \subseteq \overline{W} \cap U\,,$$

where $\overline{D}^{\overline{W}}$ denotes the closure of D in the relative topology of $\overline{W}$. We have $D = S \cap \overline{W}$ with $S \in \tau$. Let $V = S \cap W \in \mathcal{N}(x)$. This is the desired neighborhood of x.

(b) $\Longrightarrow$ (c): Let $K \subseteq X$ be compact and $U \in \tau$ such that $U \supseteq K$. For every $x \in K$ we can find $V_x \in \mathcal{N}(x)$ relatively compact such that $x \in V_x \subseteq \overline{V}_x \subseteq U$. Evidently $\{V_x\}_{x\in K}$ is an open cover of K and so using compactness we can find a finite subcover $\{V_x\}_{k=1}^n$. Then $V = \bigcup_{k=1}^n V_{x_k} \in \tau$, $\overline{V}$ is compact and $K \subseteq V \subseteq \overline{V} \subseteq U$.

(c) $\Longrightarrow$ (d): Let $\mathcal{B} = \{U \in \tau\colon \overline{U}$ is compact$\}$. Then since $\{x\}$ is compact, assertion (c) implies that $\mathcal{B}$ is a basis; see Corollary 1.1.6.

(d) $\Longrightarrow$ (a): This is obvious. □

Proposition 1.4.67. *If (X, τ) is a Hausdorff, second countable, locally compact topological space, then X has a countable basis consisting of relatively compact open sets.*

Proof. Let $\{U_n\}_{n\geq 1}$ be a basis of X. Fix $n \in \mathbb{N}$ and let $\{V_x\}_{x\in U_n}$ be an open cover of U_n such that $\overline{V}_x$ is compact and $\overline{V}_x \subseteq U_n$ for all $x \in U_n$; see Proposition 1.4.66. From Proposition 1.2.24(b) we know that U_n is second countable. So, Lindelöf's Theorem (see Theorem 1.2.27) implies that we can find a countable subcover $\{V_k^n\}_{k\geq 1}$ of U_n. Then the family $\mathcal{B} = \{V_k^n\colon n, k \in \mathbb{N}\}$ is a countable basis of X consisting of relatively compact open sets. □

The next proposition places more precisely locally compact spaces in the chart of topological spaces.

Proposition 1.4.68. *Every locally compact topological space is completely regular; see Definition* 1.2.19.

Proof. Let $x \in X$ and $C \subseteq X$ be a closed set such that $x \notin C$. Applying Proposition 1.4.66(c) yields the existence of relatively compact sets $V_1, V_2 \in \tau$ such that

$$x \in V_1 \subseteq \overline{V}_1 \subseteq V_2 \subseteq \overline{V}_2 \subseteq U = X \setminus C\,.$$

The set $\overline{V}_2$ is compact, and hence normal; see Corollary 1.4.50. Then, Urysohn's Lemma on normality (see Theorem 1.2.17) implies the existence of a continuous function $f\colon \overline{V}_2 \to [0, 1]$ such that $f|_{\overline{V}_2\setminus V_1} = 0$ and $f(x) = 1$. Let

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in \overline{V}_2\,, \\ 0 & \text{if } x \in X \setminus \overline{V}_2\,. \end{cases}$$

According to Proposition 1.1.37, $\hat{f}$ is continuous and $\hat{f}|_A = 0$ and $\hat{f}(x) = 1$. Hence, X is completely regular. □

Proposition 1.4.69. *Local compactness is preserved by continuous open surjections.*

Proof. Let X, Y be Hausdorff topological spaces with X locally compact and $f\colon X \to Y$ being a continuous, open surjection. Let $y \in Y$ and choose $x \in X$ such that $f(x) = y$. Then there exists $U \in \mathcal{N}(x)$ being relatively compact. Since f is open, $f(U) \in \mathcal{N}(y)$ and $f(\overline{U}) \subseteq Y$ is compact; see Theorem 1.4.51. Finally we have $y \in f(U) \subseteq \overline{f(U)} = f(\overline{U})$ with $f(\overline{U})$ being compact. Therefore, Y is locally compact as well. □

Of course every compact space is locally compact. In fact the following proposition is easy to prove.

Proposition 1.4.70. *If (X, τ) is a locally compact topological space, $U \in \tau$ and $C \subseteq X$ is closed, then $U \cap C$ endowed with the relative topology is locally compact.*

Proof. Let $x \in U \cap C$. Choose $V \in \mathcal{N}(x)$ relatively compact such that $x \in V \subseteq \overline{V} \subseteq U$. Then $V \cap (U \cap C)$ is a neighborhood of x in the relative topology of $U \cap C$. It holds

$$\overline{V \cap (U \cap C)}^{\tau(U \cap C)} = \overline{V} \cap (U \cap C) = \overline{V} \cap C$$

and the latter is closed in $\overline{V}$, hence compact. Therefore, $U \cap C$ is locally compact. □

Corollary 1.4.71. *Every open subset and every closed subset of a locally compact space is locally compact for the relative topology.*

We ask the natural question of when we can consider a Hausdorff topological space as a subspace of a compact topological space. Local compactness is the right concept for answering this question.

Definition 1.4.72. Let X be a Hausdorff topological space. A **compactification** of X is a compact topological space Y such that X is homeomorphic to a dense subset of Y. So we may think that X is an actual dense subset of Y.

Proposition 1.4.73. *If (X, τ) is a Hausdorff topological space and $(\hat{X}, \hat{\tau})$ a compactification of X, then X is locally compact if and only if $X \in \hat{\tau}$.*

Proof. $\Longrightarrow$: Let $x \in X$ and choose $U \in \mathcal{N}_X(x)$ relatively compact. We can find $V \in \mathcal{N}_X(x)$ such that $x \in V \subseteq U$. We have $V = W \cap X$ with $W \in \mathcal{N}_{\hat{X}}(x)$ and

$$W = W \cap \hat{X} = W \cap \overline{X} \subseteq \overline{W \cap X} = \overline{V} \subseteq \overline{U} = U \subseteq X\,.$$

This implies that x is $\hat{\tau}$-interior in X, hence $X \in \hat{\tau}$.

$\Longleftarrow$: We know that $(\hat{X}, \hat{\tau})$ is compact, hence locally compact. Since $X \in \hat{\tau}$ we conclude from Corollary 1.4.71 that X must be locally compact. □

The simplest compactification of noncompact, locally compact topological spaces is the so-called “Alexandrov one-point compactification.”

Definition 1.4.74. Let X be a Hausdorff topological space and ∞ an object not in X, called the **point at infinity**. Let $\hat{X} = X \cup \{\infty\}$ and define a topology $\hat{\tau}$ on $\hat{X}$ specifying the following open sets:
(a) $\tau \subseteq \hat{\tau}$;
(b) $\hat{X} \setminus K$ with $K \subseteq X$ compact;
(c) $\hat{X}$.
Then we say that $(\hat{X}, \hat{\tau})$ is the **one-point compactification** of X.

Theorem 1.4.75. *If $\hat{X} = X \cup \{\infty\}$ is as in Definition 1.4.74 and is endowed with the topology $\hat{\tau}$ and (X, τ) is not compact, then $(\hat{X}, \hat{\tau})$ is a compactification of X and $\hat{X}$ is Hausdorff if and only if X is locally compact.*

Proof. First we show that $(\hat{X}, \hat{\tau})$ is compact. So, let $\mathcal{L}$ be an open cover of $\hat{X}$. Then $\mathcal{L}$ must have a member U such that $\infty \in U$. Then by Definition 1.4.74, $\hat{X} \setminus U$ is compact and so it has a finite subcover $\{U_k\}_{k=1}^n \subseteq \mathcal{L}$. Evidently $\{U_k, U\}_{k=1}^n \subseteq \mathcal{L}$ is a finite open cover of $\hat{X}$ and so we conclude that $(\hat{X}, \hat{\tau})$ is compact. It is easy to see from Definition 1.4.74 that $\hat{\tau}\big|_X = \tau$, that is, the subspace topology of $X \subseteq \hat{X}$ is τ. Since X is not compact, each $\hat{\tau}$-neighborhood of ∞, $\hat{X} \setminus K$ with K compact must intersect X. Hence ∞ is a limit point of X and so $\hat{X} = \overline{X}$. This proves that $(\hat{X}, \hat{\tau})$ is a compactification of X.

Suppose now that $\hat{X}$ is Hausdorff and let $x \in X$. We can find $U, V \in \hat{\tau}$ such that $\infty \in U$, $x \in V$ and $U \cap V = \emptyset$. This implies $V \subseteq \hat{X} \setminus U = K$ with K compact; see Definition 1.4.74. Therefore, X is locally compact.

Conversely, suppose that X is locally compact. Let $x \in X$ and choose $V \in \tau$ such that $x \in V \subseteq \overline{V}$ with $\overline{V}$ compact. Let $U = \hat{X} \setminus \overline{V}$. Then $\infty \in U$, $x \in V$ and $U \cap V = \emptyset$. Hence, $\hat{X}$ is Hausdorff. ∎

Example 1.4.76. The Alexandrov compactification of $\mathbb{R}^n$ is the n-sphere $S^n = \{u \in \mathbb{R}^{n+1} : |u| = 1\}$. To see this, let $N = (0, 0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ be the **north pole**. We define the **stereographic projection** $h\colon S^n \setminus \{N\} \to \mathbb{R}^n$ by

$$h\left((u_k)_{k=1}^{n+1}\right) = \frac{(u_k)_{k=1}^{n+1}}{1 - u_{n+1}}\,.$$

This map sends a point $u \in S^n \setminus \{N\}$ to a point $x \in \mathbb{R}^n$ where the line from N to x intersects $\mathbb{R}^n$. It is a homeomorphism with inverse map

$$h^{-1}\left((x_k)_{k=1}^n\right) = \frac{\left((2x_k)_{k=1}^n, |x|^2 - 1\right)}{|x|^2 + 1}\,.$$

Therefore, $S^n \setminus \{N\}$ is homeomorphic to $\mathbb{R}^n$. Then h extends to a homeomorphism of S^n with the Alexandrov compactification $\hat{\mathbb{R}}^n$ of $\mathbb{R}^n$. We can easily visualize the stereographic projection when $n = 1$ (Fig. 1.1).

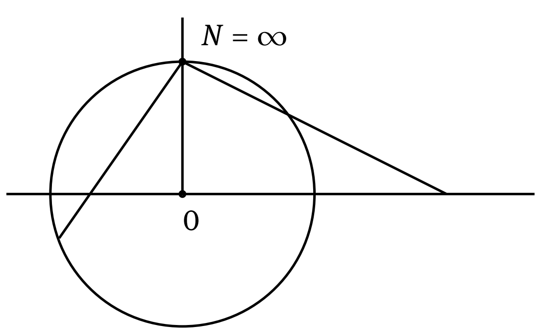

Fig. 1.1: Alexandrov one-point compactification of $\mathbb{R}^n$.

This map was known to map makers long ago. From the discussion above we see that by removing a single point from S^n we obtain a space homeomorphic to $\mathbb{R}^n$. Which point we remove is irrelevant because we can rotate any point of S^n into any other. For convenience we remove the north pole N.

Definition 1.4.77. A Hausdorff topological space X is said to be **σ-compact** if it can be expressed as the union of at most countably many compact spaces.

Proposition 1.4.78. *Let (X, τ) be a Hausdorff topological space. The following statements are equivalent:*
(a) *X is locally compact and σ-compact.*
(b) *$X = \bigcup_{k\geq 1} U_k$ with U_k open, relatively compact such that $\overline{U}_k \subseteq U_{k+1}$ with $k \in \mathbb{N}$.*
(c) *X is locally compact and Lindelöf.*

Proof. (a) $\Longrightarrow$ (b): By hypothesis we have $X = \bigcup_{k\geq 1} K_k$ with $K_k \subseteq X$ compact. Proposition 1.4.66(c) says that we can find $U_1 \supseteq K_1$ open and relatively compact. By induction we can find U_k open, relatively compact such that $U_k \supseteq \overline{U}_{k-1} \cup K_k$. Then $\{U_k\}_{k\geq 1}$ is the desired sequence of open sets.

(b) $\Longrightarrow$ (c): Let $\mathcal{L} = \{U_i\}_{i\in I}$ be an open cover of X. For each $m \in \mathbb{N}$ we can find a finite subfamily $\{U_m^k\}_{k=1}^{n(m)} \subseteq \mathcal{L}$ that covers $\overline{U}_i$ = compact. The family $\{U_m^k : 0 \leq k \leq n(m), m \in \mathbb{N}\} \subseteq \mathcal{L}$ is a countable subcover; thus X is Lindelöf.

(c) $\Longrightarrow$ (a): Let $\mathcal{L} = \{U_x\}_{x\in X}$ be a cover by relatively compact open sets; see Proposition 1.4.66(c). The Lindelöf property implies that we can extract a countable subcover. Therefore, X is σ-compact. $\square$

We introduce a generalization of σ-compactness that is determined by some requirement on the behavior of their coverings.

Definition 1.4.79. Let X be a Hausdorff topological space.
(a) Given two covers $\mathcal{L} = \{U_i\}_{i\in I}$ and $\mathcal{L}' = \{V_j\}_{j\in J}$ of X. We say that $\mathcal{L}$ is a **refinement** of $\mathcal{L}'$ if for each $i \in I$ there is a $j \in J$ such that $U_i \subseteq V_j$. We write $\mathcal{L} \prec \mathcal{L}'$.
(b) We say that a cover $\mathcal{L} = \{U_i\}_{i\in I}$ of X is **locally finite** if for every $x \in X$ there exists $V \in \mathcal{N}(x)$ that intersects a finite number of U_i's.
(c) We say that the cover $\mathcal{L} = \{U_i\}_{i\in I}$ of X is **point finite** if for every $x \in X$ there are at most finitely many indices $i \in I$ such that $x \in U_i$.

Remark 1.4.80. Given two covers $\mathcal{L} = \{U_i\}_{i\in I}$ and $\mathcal{L}' = \{V_j\}_{j\in J}$ of X we can define $\mathcal{L}_0 = \{U_i \cap V_j : (i, j) \in I \times J\}$, which is also a cover of X refining both $\mathcal{L}$ and $\mathcal{L}'$. Moreover, if both $\mathcal{L}$ and $\mathcal{L}'$ are locally finite (resp. point finite), then so is $\mathcal{L}_0$. A common refinement of both $\mathcal{L}$ and $\mathcal{L}'$ is also a refinement of $\mathcal{L}_0$.

A refinement of a cover may contain more elements than the given cover.

Definition 1.4.81. A refinement $\mathcal{L} = \{U_i\}_{i\in I}$ of the cover $\mathcal{L}' = \{V_j\}_{j\in J}$ is said to be **precise** if $I = J$ and $U_i \subseteq V_i$ for all $i \in I$.

Proposition 1.4.82. *If X is a Hausdorff topological space and the cover $\mathcal{L}' = \{V_j\}_{j\in J}$ of X has a locally finite (resp. point finite) refinement $\mathcal{L} = \{U_i\}_{i\in I}$, then it has a precise locally finite (resp. point finite) refinement $\hat{\mathcal{L}} = \{\hat{U}_j\}_{j\in J}$. Moreover, if $\mathcal{L}$ is open, then so is $\hat{\mathcal{L}}$.*

Proof. Let $\xi\colon I \to J$ be the map that assigns to each $i \in I$ a $j \in J$ such that $U_i \subseteq V_j$; see Definition 1.4.79(a). For every $j \in J$ let $\hat{U}_j = \bigcup\{U_i\colon \xi(i) = j\}$ (some $\hat{U}_j$ may be empty). Then $\hat{U}_j \subseteq V_j$ for every $j \in J$ and $\hat{\mathcal{L}} = \{\hat{U}_j\}_{j\in J}$ is a cover of X. Clearly, $\hat{\mathcal{L}}$ is locally finite (resp. point finite) if $\mathcal{L}$ is and it is open if $\mathcal{L}$ is open. □

Definition 1.4.83. A Hausdorff topological space X is said to be **paracompact** if each open cover of X admits a locally finite refinement.

An immediate consequence of this definition is the following result.

Proposition 1.4.84. *Every compact topological space is paracompact.*

Closely related to paracompactness is the notion of **partition of unity**, which is essentially a **variable convex combination**.

Definition 1.4.85. Let X be a Hausdorff topological space and $f\colon X \to \mathbb{R}$ a function.
(a) The **support** of f is the closed set $\operatorname{supp} f := \overline{\{x \in X\colon f(x) \neq 0\}}$.
(b) A **partition of unity** on X is a family $\{f_i\}_{i\in I}$ of continuous functions $f_i\colon X \to [0, 1]$ such that
   (i) $\{\operatorname{supp} f_i\}_{i\in I}$ form a locally finite closed cover of X;
   (ii) $\sum_{i\in I} f_i(x) = 1$ (the sum is well-defined because of (i)).
   If $\mathcal{L}' = \{V_j\}_{j\in J}$ is an open cover of X, then we say that a partition of unity $\{f_j\}_{j\in J}$ is subordinated to $\mathcal{L}'$ if $\operatorname{supp} f_j \subseteq V_j$ for each $j \in J$.

There is a close relation between paracompactness and partition of unity. The proof of the following theorem is very technical and so it is omitted. We refer to Dugundji [91, Theorem 4.2, p. 170].

Theorem 1.4.86. *A Hausdorff topological space is paracompact if and only if every open cover on X admits a locally finite partition of unity subordinated to the open cover.*

This theorem allows us to fix the place of paracompactness in the chart of topological spaces.

Proposition 1.4.87. *Every paracompact space is normal.*

Proof. Let C_1 and C_2 be two disjoint, closed subspaces of X. We consider the open cover $\mathcal{L} = \{X \setminus C_1, X \setminus C_2\}$. Then Theorem 1.4.86 implies that there is a partition of unity $\{f_1, f_2\}$ subordinated to $\mathcal{L}$. Then $f_1|_{C_2} = 1$ and $f_1|_{C_1} = 0$ and so by Urysohn's Normality Lemma (see Theorem 1.2.17) we conclude that X is normal. □

Closing this section, we mention that there is a "locally compact" version of the Tietze Extension Theorem; see Theorem 1.2.44. This version of the Tietze result reads as follows; see Hewitt–Stromberg [145, Theorem 7.40, p. 99].

Theorem 1.4.88. *If X is locally compact, $K \subseteq X$ is a nonempty, compact set and $U \subseteq K$ is open and $K \subseteq U$, then for every $f \in C(K, \mathbb{R})$ there exists $\hat{f} \in C(X, \mathbb{R})$ with compact support such that $\hat{f}|_K = f$ and f vanishes on $X \setminus U$.*

1.5 Metric Spaces – Baire Category

Metric spaces are a very important class of topological spaces. In fact the development of metric spaces led to the more general notion of topological space. In metric spaces the metric leads to an analysis that is primarily based in the properties of the real line.

Definition 1.5.1. Let X be a set. A **metric** on X is a map $d\colon X \times X \to \mathbb{R}$ such that the following hold:
(a) $d(x, u) = 0$ if and only if $x = u$;
(b) $d(x, u) = d(u, x)$ for all $x, u \in X$ (symmetry);
(c) $d(x, u) \leq d(x, v) + d(v, u)$ for all $x, u, v \in X$ (triangle inequality).
The pair (X, d) of a set X and of a metric d on X is said to be a **metric space**. If d does not satisfy (a), then d is called a **semimetric** (in French "ecart") and (X, d) is a **semimetric space**.

Remark 1.5.2. If d is a metric, then, based on (a)–(c), it is clear that $d(x, y) \geq 0$ for all $x, y \in X$. If d is a semimetric and $\sim$ is the equivalence relation defined by $x \sim u$ if and only if $d(x, u) = 0$, then $X/\sim$ is a metric space with metric $\hat{d}([x], [u]) = d(x, u)$. Here, for $x \in X$, $[x]$ is the corresponding equivalence class.

Definition 1.5.3. (a) Let (X, d) be a metric space and $A \subseteq X$. The **diameter** of A is defined by

$$\operatorname{diam} A = \sup[d(x, u)\colon x, u \in A] .$$

If $\operatorname{diam} A < \infty$, then we say that A is **bounded**. Otherwise A is **unbounded**. When $\operatorname{diam} X < \infty$, then we say that d is a **bounded metric**. In addition, for $x \in X$ and $r > 0$, the **open ball** with center x and radius r is defined by

$$B_r(x) = \{u \in X\colon d(u, x) < r\} .$$

The corresponding **closed ball** with center x and radius r is defined by

$$\overline{B}_r(x) = \{u \in X\colon d(u, x) \leq r\} .$$

(b) Let (X, d) be a metric space. A set $A \subseteq X$ is said to be d**-open** (or simply **open**) if for every $x \in A$ we can find $r = r(x) > 0$ such that $B_r(x) \subseteq A$. The collection

$$\tau_d = \{A \subseteq X\colon A \text{ is } d\text{-open}\}$$

is a topology on X called the **metric topology** on (X, d).
(c) A topological space (X, τ) is said to be **metrizable** if $\tau = \tau_d$ for some metric d on X. This metric is then said to be **compatible** with the topology. If for two metrics d_1 and d_2 on X, we have $\tau_{d_1} = \tau_{d_2}$, then we say that d_1 and d_2 are **equivalent**.

Remark 1.5.4. The distinction between metric and metrizable spaces is a subtle one. In the case of a metric space we already have a fixed metric. For a metrizable space we have not yet decided from the multitude of equivalent metrics. Note that if d is compatible, then so is kd with $k \in \mathbb{N}$ or $\hat{d}(x,u) = (d(x,u))(1 + d(x,u))$ and $\hat{d}_0(x,u) = \min\{1, d(x,u)\}$. The last two metrics are bounded even if d is not. From the triangle inequality we have

$$|d(x,u) - d(y,v)| \le d(x,y) + d(u,v) \quad \text{for all } x,u,y,v \in X\,. \tag{1.5.1}$$

It follows that d is jointly continuous. Of course τ_d is Hausdorff and first countable and $u_n \xrightarrow{\tau_d} u$ if and only if $d(u_n, u) \to 0$.

In Proposition 1.2.22 we saw that second countability implies separability. For metrizable spaces the two notions are equivalent.

Proposition 1.5.5. *A metrizable space is second countable if and only if it is separable.*

Proof. $\Longrightarrow$: This follows from Proposition 1.2.22.

$\Longleftarrow$: Let (X,τ) be a separable metrizable space and d a compatible metric, that is, $\tau_d = \tau$. Let $D \subseteq X$ be a countable dense set and consider the collection $\mathcal{L} = \{B_{1/n}(x) \colon x \in D, n \in \mathbb{N}\}$. Clearly, $\mathcal{L}$ is a countable basis for the topology τ; see Corollary 1.1.6. ☐

Combining this proposition with Proposition 1.2.24(b) we have the following result.

Corollary 1.5.6. *If X is a separable metrizable space and $A \subseteq X$, then A is separable.*

Definition 1.5.7. Let (X,τ) be a topological space. A set A is said to be an F_σ**-set** if it is the union of at most countably many closed sets. A set C is said to be a G_δ**-set** if it is the intersection of at most countably many open sets.

Proposition 1.5.8. *If X is a metrizable space, then every closed set is G_δ and every open set is F_σ.*

Proof. Let $C \subseteq X$ be closed. Then $U_n = \{x \in X \colon d(x,C) < 1/n\}$ is open because of the continuity of d. Furthermore $C = \bigcap_{n\ge1} U_n$. So C is G_δ. Next let $U \subseteq X$ be open. Since $X \setminus U$ is closed, the first part yields that $X \setminus U = \bigcap_{n\ge1} U_n$ with U_n open. Hence, $U = \bigcup_{n\ge1}(X \setminus U_n)$ and so U is F_σ. ☐

Definition 1.5.9. (a) Let (X,d) be a metric space. A sequence $\{x_n\}_{n\ge1} \subseteq X$ is said to be a **Cauchy sequence** if for any given $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \ge 1$ such that $d(x_n, x_m) \le \varepsilon$ for all $n, m \ge n_0$, that is, $d(x_n, x_m) \to 0$ as $n, m \to +\infty$. We say that (X,d) is **complete** if every Cauchy sequence in X converges in X.

(b) Let (X,τ) be a topological space. We say that X is **topologically complete** if there is a compatible complete metric d, that is, $\tau_d = \tau$.

Remark 1.5.10. The property of completeness is metric dependent. So it can happen that two metrics are equivalent, that is, they generate the same topology, but one is complete and the other not. On the other hand, topological completeness is a topological property.

Example 1.5.11. The interval $(-1, 1)$ with the usual metric is not a complete metric space but it is topologically complete since it is homeomorphic to $\mathbb{R}$, which is complete. The function $h\colon (-1, 1) \to \mathbb{R}$ defined by $h(x) = x/(1 - x^2)$ for all $x \in (-1, 1)$ is a homeomorphism between the two spaces.

Definition 1.5.12. Let (X, d) and (Y, ρ) be two metric spaces. A map $f\colon X \to Y$ is said to be an **isometry** if $d(x, u) = \rho(f(x), f(u))$ for all $x, u \in X$. If f is a surjective isometry, then we say that X and Y are **isometric** spaces. Otherwise we say that f is an isometric embedding.

Remark 1.5.13. Thus an isometric surjection is a distance preserving homeomorphism. In the case of an isometric embedding $f\colon X \to Y$ we may think of X as a subspace of Y.

Every metric space can be isometrically and densely embedded in a complete metric space.

Theorem 1.5.14. *If (X, d) is any metric space, then there is a complete metric space (Y, ρ) and an isometry $f\colon X \to Y$ such that $f(X)$ is dense in Y. We say that Y is the **completion** of X.*

Proof. Let $f_x(u) = d(x, u)$ for all $x, u \in X$. Choose a point $v \in X$ and let

$$S(X, d) = \{f_v + h\colon h \in C_{\mathrm{b}}(X, \mathbb{R})\} \ .$$

On $S(X, d)$ we consider the supremum metric d_∞ defined by

$$d_\infty(f_v + h, f_v + \hat{h}) = \sup\left[\left|h(x) - \hat{h}(x)\right| : x \in X\right] \ .$$

For any $x, u, y \in X$ we have

$$|d(x, y) - d(u, y)| \le d(x, u)$$

(see (1.5.1)) and equality holds if $y = x$ or $y = u$. Therefore, for any $u \in X$, taking $x = v$, we have

$$f_u - f_x \in C_{\mathrm{b}}(X, \mathbb{R}) \ , \quad d_\infty(f_x, f_u) = d(x, u) \ .$$

In addition we have $f_u \in S(X, d)$ and $S(X, d)$ does not depend on the choice of $v \in X$. Hence, the map $x \to f_x$ from X into $S(X, d)$ is an isometry for d and d_∞. Let Y be the d_∞-closure of the range of this map into $S(X, d)$. But $(C_{\mathrm{b}}(X, \mathbb{R}), d_\infty)$ is complete; recall that the uniform limit of continuous functions is continuous. Hence (Y, d_∞) is complete and this is the completion of (X, d). □

Now we can provide a necessary and sufficient condition for the completeness of a metric space. The necessary part of the result is known as “Cantor’s Intersection Theorem.”

Theorem 1.5.15. *A metric space (X, d) is complete if and only if every decreasing sequence $\{C_n\}_{n\ge1}$ of nonempty, closed subsets of X such that $\operatorname{diam} C_n \to 0$ as $n \to \infty$, has a singleton intersection.*

Proof. $\Longrightarrow$: Let $C = \bigcap_{n\geq1} C_n$. Then $\operatorname{diam} C \leq \operatorname{diam} C_n$ for all $n \in \mathbb{N}$. Hence, $\operatorname{diam} C = 0$. This means that C is empty or a singleton. We show that $C \neq \emptyset$. For each $n \in \mathbb{N}$ we pick $u_n \in C_n$. Then for $n \geq m$ we have $d(u_n, u_m) \leq \operatorname{diam} C_m \to 0$ as $m \to \infty$. So $\{u_n\}_{n\geq1} \subseteq X$ is a Cauchy sequence and the completeness of X implies that there exists $u \in X$ such that $u_n \to u$. Evidently $u \in C$ and so $C = \bigcap_{n\geq1} C_n = \{u\}$.

$\Longleftarrow$: Let $\{u_n\}_{n\geq1} \subseteq X$ be a Cauchy sequence. Set $C_n = \overline{\{u_k : k \geq n\}}$. Since $\{u_n\}_{n\geq1}$ is a Cauchy sequence, we have $\operatorname{diam} C_n \to 0$. By hypothesis $\bigcap_{n\geq1} C_n = \{u\}$ and so we have $u_n \to u$ in X, which means that X is complete. □

Now we consider the Cartesian product of metric spaces. To this end, let $\{X_n\}_{n\geq1}$ be a sequence of nonempty Hausdorff topological spaces and let $X = \prod_{n\geq1} X_n$ be furnished with the product topology.

Proposition 1.5.16. *The product topology on $X = \prod_{n\geq1} X_n$ is metrizable if and only if the space X_n is metrizable for each $n \in \mathbb{N}$.*

Proof. $\Longrightarrow$: Let d be a compatible metric for X. For each $n \in \mathbb{N}$ we fix a $y_n \in X_n$. Then for $u \in X_m$ we define $\hat{u} = (u_k)_{k\geq1} \in X$ by setting $u_k = y_k$ for $k \neq m$ and $u_m = u$. Now we define a metric d_m on X_m by setting $d_m(u, v) = d(\hat{u}, \hat{v})$. It is easy to see that d_m is indeed a metric on X_m. Note that d-convergence in X is equivalent to componentwise convergence. From this it follows easily that τ_{d_m} coincides with the topology of X_m.

$\Longleftarrow$: Assume that each X_n is metrizable and let d_n be a compatible metric. We define a metric d on the product X by setting

$$d((u_n), (v_n)) = \sum_{n\geq1} \frac{1}{2^n} \frac{d_n(u_n, v_n)}{1 + d_n(u_n, v_n)} .$$

It is straightforward that d is a metric. Let $\{\hat{u}_\alpha\}_{\alpha\in J} = \{(u_n^\alpha)\}_{\alpha\in J} \subseteq X$ be a net. We have

$$d(\hat{u}_\alpha, \hat{u}) \to 0 \text{ with } \hat{u} = (u_n) \text{ if and only if } \lim_{\alpha\in J} d_n(u_n^\alpha, u_n) = 0 , \tag{1.5.2}$$

for all $n \in \mathbb{N}$. From (1.5.2) we infer that the product topology and the τ_d-topology on X coincide. □

In a similar fashion we can also have the following result.

Proposition 1.5.17. *The product topology on X is topologically complete if and only if the space X_n is topologically complete for each $n \in \mathbb{N}$.*

Proposition 1.5.18. *If $\{X_n\}_{n\geq1}$ is a sequence of metrizable spaces and $X = \prod_{n\geq1} X_n$, then X is separable if and only if X_n is separable for each $n \in \mathbb{N}$.*

Proof. $\Longrightarrow$: This is a consequence of the fact that the continuous image of a separable space is separable as well; see Proposition 1.2.24(c). In our case the continuous map is the projection to the nth factor.

$\Longleftarrow$: From the proof of Proposition 1.5.16 we know that the product topology on X is generated by the metric

$$d(\hat{u}, \hat{v}) = \sum_{n\geq 1} \frac{1}{2^n} \frac{d_n(u_n, v_n)}{1 + d_n(u_n, v_n)} \quad \text{for all } \hat{u} = (u_n), \hat{v} = (v_n) \in X \,.$$

For each $n \in \mathbb{N}$ let D_n be a countable, dense subset of X_n. Fix $u_n \in D_n$ for each $n \in \mathbb{N}$ and consider the set $D \subseteq X$ defined by

$$D = \{(y_n) \in X \colon y_n \in D_n \text{ for each } n \in \mathbb{N} \text{ and } y_n = u_n \text{ eventually}\} \,.$$

Evidently $D \subseteq X$ is countable and dense. Therefore X is separable. □

Definition 1.5.19. The **Hilbert cube** is the space $\mathbb{H} = [0, 1]^{\mathbb{N}}$, that is, the space of all real sequences with values in $[0, 1]$.

Remark 1.5.20. Evidently $\mathbb{H}$ is topologically complete, separable, and compact, which follows from the Propositions 1.5.17 and 1.5.18 as well as Theorem 1.4.56.

The next theorem, known as “Urysohn’s Theorem,” says that in a sense $\mathbb{H}$ is the canonical separable metrizable space.

Theorem 1.5.21 (Urysohn’s Theorem). *Every separable metrizable space is homeomorphic to a subset of* $\mathbb{H}$.

Proof. Let (X, d) be a separable metric space and $D = \{y_n\}_{n\geq 1}$ a countable dense subset. We define $\xi_n(u) = \min\{1, d(u, y_n)\}$ for all $n \in \mathbb{N}$ and consider $\xi \colon X \to \mathbb{H}$ defined by $\xi(u) = (\xi_n(u))_{n\geq 1}$ for all $u \in X$. Each ξ_n is continuous, hence so is ξ. Suppose that $\xi(u) = \xi(v)$ and let $\{y_{n_k}\}_{k\geq 1} \subseteq \{y_n\}_{n\geq 1}$ such that $y_{n_k} \to u$. We have $\lim_{k\to\infty} d(v, y_{n_k}) = 0$, hence $d(v, u) = 0$, which means that $u = v$ and so ξ is $1-1$. Finally we need to show that ξ^{-1} is continuous. To this end, let $\xi(v_n) \to \xi(v)$. Pick $\varepsilon > 0$ and u_m such that $d(v, u_m) < \varepsilon$. Note that

$$d(v_n, u_m) \to d(v, u_m) \quad \text{as } n \to \infty \,,$$

which means $d(v_n, u_m) < \varepsilon$ for all $n \geq n_0$. Hence, by the triangle inequality we derive $d(v_n, v) < 2\varepsilon$ for all $n \geq n_0$. Therefore, $v_n \to v$ and so ξ^{-1} is continuous. □

Some features of metrizable spaces are not topological and depend on the particular compatible metric. Such are Cauchy sequences (see Definition 1.5.9(a)) and uniform continuity, which we are about to introduce.

Definition 1.5.22. Let (X, d) and (Y, ρ) be two metric spaces and $f \colon X \to Y$ a map.

(a) We say that f is **uniformly continuous** if for every given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$d(x, u) < \delta \quad \text{implies} \quad \rho(f(x), f(u)) < \varepsilon \quad \text{for all } x, u \in X \,.$$

(b) We say that f is k-**Lipschitz** if

$$\rho(f(x), f(u)) \le kd(x, u) \quad \text{for all } x, u \in X \text{ with } k > 0\,.$$

Remark 1.5.23. A continuous function need not be uniformly continuous. For example, the function $f(x) = x^2$ for $x \in \mathbb{R}$ is continuous but not uniformly continuous. Indeed, note that for $\varepsilon > 0$ the $\delta > 0$ gets smaller as $|x|$ increases. A k-Lipschitz map is uniformly continuous. A 1-Lipschitz map is called **nonexpansive** and if $k \in (0, 1)$ we say that f is a **contraction**.

Proposition 1.5.24. *If (X, d) is a metric space and $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuous satisfying*
(a) *φ is nondecreasing, that is, $x \le u$ implies $\varphi(x) \le \varphi(u)$ for all $x, u \ge 0$;*
(b) *φ is subadditive, that is, $\varphi(x + u) \le \varphi(x) + \varphi(u)$ for all $x, u \ge 0$;*
(c) *$\varphi(x) = 0$ if and only if $x = 0$,*
then $\varphi \circ d$ is a metric on X and the identity maps

$$i_1\colon (X, d) \to (X, \varphi \circ d) \quad \text{and} \quad i_2\colon (X, \varphi \circ d) \to (X, d)$$

are both uniformly continuous.

Proof. Applying (a)–(c) it is straightforward to check that $\varphi \circ d$ is a metric on X. Moreover, for given $\varepsilon > 0$ there exists $\delta > 0$ such that $0 \le t < \delta$ implies $0 \le \varphi(t) < \varepsilon$ as well as $0 \le \varphi(t) < \eta = \varphi(\varepsilon)$ implies $0 \le t < \delta$. Here we have used the continuity and monotonicity of φ. Thus we have uniform continuity for both i_1 and i_2. □

Proposition 1.5.25. *If (X, d) and (Y, ρ) are two metric spaces and $f\colon X \to Y$ is uniformly continuous, then f maps Cauchy sequences in X to Cauchy sequences in Y.*

Proof. Let $\{u_n\}_{n\ge1}$ be a Cauchy sequence in X, and for $\varepsilon > 0$ choose $\delta = \delta(\varepsilon) > 0$ such that $d(x, v) < \delta$ implies $\rho(f(x), f(v)) < \varepsilon$ for all $x, v \in X$.

Let $B \subseteq X$ be a ball of radius less than $\delta/2$, which contains $\{u_n\}_{n\ge n_0}$ for some $n_0 \in \mathbb{N}$. Then $f(B)$ contains $\{f(u_n)\}_{n\ge n_0}$. Note that $\operatorname{diam} B < \delta$. Hence $\operatorname{diam} f(B) < \varepsilon$. Thus $f(B)$ is included in a ball $D \subseteq Y$ of radius $\varepsilon > 0$ and so $D \supseteq \{f(u_n)\}_{n\ge\hat{n}}$ for some $\hat{n} \in \mathbb{N}$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\{f(u_n)\}_{n\in\mathbb{N}} \subseteq Y$ is a ρ-Cauchy sequence. □

Remark 1.5.26. The result above fails if f is only continuous. To see this consider the function $f(x) = 1/x$ for all $x \in (0, 1)$, which is continuous but not uniformly continuous. Let $u_n = 1/n$ with $n \in \mathbb{N}$. This is a Cauchy sequence in $(0, 1)$ but $f(u_n) = n$, which is not a Cauchy sequence.

Theorem 1.5.27. *If (X, d) is a metric space, $D \subseteq X$ a set, (Y, ρ) is a complete metric space and $f\colon D \to Y$ is uniformly continuous, then there exists a unique uniformly continuous map $\hat{f}\colon \overline{D} \to Y$ such that $\hat{f}|_D = f$. In particular, if $Y = \mathbb{R}$ then $\sup_D |f| = \sup_{\overline{D}} |\hat{f}|$.*

Proof. Let $\tilde{u} \in \overline{D}$. Then we find a sequence $\{u_n\}_{n\ge1} \subseteq D$ such that $u_n \to \tilde{u}$ in (X, d). The sequence $\{u_n\}_{n\ge1}$ is a d-Cauchy sequence and then $\{f(u_n)\}_{n\ge1} \subseteq Y$ is a ρ-Cauchy sequence because of Proposition 1.5.25. The completeness of Y implies that $f(u_n) \to y \in Y$. This y is independent of the particular sequence in D approaching $\tilde{u} \in \overline{D}$. Indeed, let $\{x_n\}_{n\geq 1} \subseteq D$ be another sequence such that $x_n \to \tilde{u}$ in (X, d). We define

$$h_n = \begin{cases} x_n & \text{if } n = \text{odd} \\ u_n & \text{if } n = \text{even} \end{cases} \quad \text{with} \quad n \in \mathbb{N} .$$

We see that $h_n \to \tilde{u}$ and then $f(h_n) \to y$. Note that $\{f(h_n)\}_{n\geq 1}$ is a Cauchy sequence and for the subsequence $\{f(u_n)\}_{n\geq 1}$ we have that it converges to y in (Y, ρ). Hence, we have shown that y is independent of the sequence $u_n \to \tilde{u} \in \overline{D}$. Therefore, we can set $\hat{f}(\tilde{u}) = y$.

Now we show that $\hat{f}$ is uniformly continuous. From the uniform continuity of f we know that for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x, u) < \delta \quad \text{implies} \quad \rho(f(x), f(u)) < \varepsilon \quad \text{for all } x, u \in D . \tag{1.5.3}$$

Suppose $x, v \in \overline{D}$ with $d(x, v) < \delta$. Then there exist $\{x_n\}_{n\geq 1}, \{u_n\}_{n\geq 1} \subseteq D$ such that $x_n \to x$ and $v_n \to v$ in (X, d). Hence, $d(x_n, v_n) \to d(x, v)$ and so $d(x_n, v_n) < \delta$ for all $n \geq n_0$. Taking (1.5.3) into account we conclude that $\rho(f(x_n), f(v_n)) < \varepsilon$ for all $n \geq n_0$. Hence, $\rho(f(x), f(v)) \leq \varepsilon$. This proves the uniform continuity of the extension $\hat{f}$. Clearly this extension is unique and we have $\sup_D |f| = \sup_{\overline{D}} |\hat{f}|$. ☐

Definition 1.5.28. Let (X, d) be a metric space. Recall that

$$C_b(X, \mathbb{R}) = \{f : X \to \mathbb{R} \mid f \text{ is bounded and continuous}\} .$$

We also introduce the subspace

$$U_b(X, \mathbb{R}) = \{f : X \to \mathbb{R} \mid f \text{ is bounded and uniformly continuous}\}$$

of $C_b(X, \mathbb{R})$. On them we consider the **supremum metric** defined by

$$d_\infty(f, g) = \sup_{x\in X} |f(x) - g(x)| .$$

Remark 1.5.29. If X is a metrizable space and d, e are two compatible metrics, then in general we have $U_d(X, \mathbb{R}) \neq U_e(X, \mathbb{R})$. For example, the function $x \to 1/x$ on $(0, 1)$ is not uniformly continuous for the usual metric on $(0, 1)$, but it is uniformly continuous for the metric $\rho(x, u) = |1/x - 1/u|$ for all $x, u \in (0, 1)$.

Proposition 1.5.30. *If (X, d) is a metric space, then X is isometrically embedded into $U_d(X, \mathbb{R})$.*

Proof. We fix $u_0 \in X$ and then for each $x \in X$, let $\eta_x : X \to \mathbb{R}$ be the function defined by $\eta_x(u) = d(x, u) - d(u_0, u)$ for all $u \in X$. We have

$$|\eta_x(u) - \eta_x(v)| \leq |d(x, u) - d(x, v)| + |d(u_0, u) - d(u_0, v)| \leq 2d(u, v) ,$$

which shows that η_x is 2-Lipschitz. In addition we have $\eta_x(u) \le d(x, u_0)$ for all $u \in X$. Thus, η_x is bounded. Consequently we have $\eta_x \in U_d(X, \mathbb{R})$. Note that

$$|\eta_x(u) - \eta_v(u)| \le d(x, v) \quad \text{for all } u \in X\,,$$

implying $d_\infty(\eta_x, \eta_v) \le d(x, v)$. Moreover, we have $|\eta_x(v) - \eta_v(v)| = d(x, v)$. Therefore, $d_\infty(\eta_x, \eta_v) = d(x, v)$, which means that $x \to \eta_x$ is an isometry. This proves that X is isometrically embedded into $U_d(X, \mathbb{R})$. $\square$

Now we turn our attention to compact metric spaces.

Definition 1.5.31. Let (X, d) be a metric space and $\varepsilon > 0$. An **ε-net** in X is a finite set A in X such that $X = \bigcup_{a \in A} B_\varepsilon(a)$. That is, for every $x \in X$ there exists $a \in A$ such that $d(x, a) < \varepsilon$. We say that (X, d) is **totally bounded** if for every $\varepsilon > 0$ it has an ε-net.

Remark 1.5.32. Clearly a compact metric space is totally bounded.

Proposition 1.5.33. *If the metric space (X, d) is totally bounded, then it is separable.*

Proof. For each $n \in \mathbb{N}$, let $A_n \subseteq X$ be a finite set such that $X = \bigcup_{x \in A_n} B_{1/n}(x)$. Let $D = \bigcup_{n \ge 1} A_n$. Then D is countable and dense in X. $\square$

Proposition 1.5.34. *If (X, d) is a sequentially compact metric space and let $\mathcal{L}$ be an open cover of X, then there is a $\delta > 0$ such that every $A \subseteq X$ with $\operatorname{diam} A < \delta$ is contained in some $U \in \mathcal{L}$.*

Proof. Arguing by contradiction, suppose that we cannot find such a $\delta > 0$. Then for every $n \in \mathbb{N}$ choose $A_n \subseteq X$ with $\operatorname{diam} A_n < 1/n$ and A_n is not contained in any $U \in \mathcal{L}$. Choose $x_n \in A_n$. Since X is sequentially compact, by passing to a subsequence if necessary, we may assume that $x_n \to x$. Let $U \in \mathcal{L} \cap \mathcal{N}(x)$ and choose $\varrho > 0$ such that $B_\varrho(x) \subseteq U$. Then $x_n \in B_{\varrho/2}(x)$ for all $n \ge n_0$ with $1/n_0 < \varrho/2$. Since $\operatorname{diam} A_{n_0} < 1/n_0 < \varrho/2$, we have $A_{n_0} \subseteq B_\varrho(x) \subseteq U$, a contradiction. This proves the proposition. $\square$

Remark 1.5.35. A $\delta > 0$ satisfying the property above is called the **Lebesgue number** of the cover $\mathcal{L}$.

The next theorem provides a complete characterization of compact metric spaces.

Theorem 1.5.36. *If (X, d) is a metric space, then the following statements are equivalent:*
(a) *X is compact;*
(b) *X is complete and totally bounded;*
(c) *X is sequentially compact.*

Proof. (a) $\Longrightarrow$ (b): Since (X, d) is compact, every Cauchy sequence $\{x_n\}_{n \ge 1}$ has a cluster point $x \in X$, see Remark 1.4.58. We claim that $x_n \to x$ in X. Since $\{x_n\}_{n \ge 1}$ is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ for every given $\varepsilon > 0$ such that

$$d(x_n, x_m) < \varepsilon \quad \text{for all } n, m \ge n_0\,. \tag{1.5.4}$$

Since x is a cluster point of the Cauchy sequence, we can find $k \geq n_0$ such that

$$d(x_k, x) < \varepsilon \, . \tag{1.5.5}$$

Then, combining (1.5.4) and (1.5.5), we have for $n \geq n_0$

$$d(x_n, x) \leq d(x_n, x_k) + d(x_k, x) < 2\varepsilon \, ,$$

which means that $x_n \to x$ in X and so X is complete.

For every $\varepsilon > 0$ we have $X = \bigcup_{x \in X} B_\varepsilon(x)$. The compactness of X implies that we can find $x_1, \ldots, x_m$ such that $X = \bigcup_{n=1}^m B_\varepsilon(x_n)$. Thus, X is totally bounded.

(b) $\Longrightarrow$ (c): Let $\{x_n\}_{n \geq 1}$ be a sequence in X. Since X is totally bounded, a subsequence S_1 of $\{x_n\}_{n \geq 1}$ must be in a set $B_1 = \{u \in X \colon d(y_1, u) < 1\}$. Evidently, B_1 is totally bounded. Hence, there exists a subsequence S_2 of S_1, which will be in $B_2 = \{u \in B_1 \colon d(y_2, u) < 1/2\}$. By induction for each $n \in \mathbb{N}$ we can have a subsequence S_{n+1} of S_n, which is in $B_{n+1} = \{u \in X \colon d(y_{n+1}, u) < 1/(n+1)\}$. Let $i_1 < i_1 < \ldots < i_n < \ldots$ be such that $x_{i_n} \in S_n$. Then $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy sequence and thus converges. This proves that X is sequentially compact.

(c) $\Longrightarrow$ (a): Let $\mathcal{L}$ be an open cover of X and let $\delta > 0$ be the Lebesgue number of $\mathcal{L}$; see Proposition 1.5.34 and Remark 1.5.35. First we show that X is totally bounded. If this is not the case, then we can find $\varepsilon > 0$ such that no finite family of balls of radius $\varepsilon > 0$ cover X. Inductively we can generate a sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that for all $n \in \mathbb{N}$, $x_n \notin \bigcup_{k<n} B_\varepsilon(x_k)$. For $n \neq m$, we have $d(x_n, x_m) \geq \varepsilon$. This sequence cannot have a convergent subsequence and this contradicts the hypothesis that X is sequentially compact. Therefore X is totally bounded. We choose a $\delta/3$-net $\{x_n\}_{n=1}^m$. For each $n \leq m$ let $U_n \in \mathcal{L}$ such that $B_{\delta/3}(x_n) \subseteq U_n$. Then $\{U_n\}_{n=1}^m$ is a finite subcover of $\mathcal{L}$ and this proves the compactness of X. ☐

Corollary 1.5.37. *A metric space is totally bounded if and only if its completion is compact.*

Since bounded sets in $\mathbb{R}^N$ are totally bounded we can state the following characterization of compact sets in $\mathbb{R}^N$. The result is known as the “Heine–Borel Theorem.”

Theorem 1.5.38 (Heine–Borel Theorem). *A set $C \subseteq \mathbb{R}^N$ is compact if and only if C is closed and bounded.*

Proposition 1.5.39. *If (X, d) and (Y, ρ) are metric spaces with X being sequentially compact and if $f \colon X \to Y$ is continuous, then f is uniformly continuous.*

Proof. Given $\varepsilon > 0$ and $x \in X$, let $V_x = f^{-1}(B_{\varepsilon/2}(f(x))) \in \mathcal{N}(x)$. Then, for $u, v \in V_x$ we have

$$\rho(f(u), f(v)) < \varepsilon \, . \tag{1.5.6}$$

We know that X is sequentially compact because of Theorem 1.5.36. By Proposition 1.5.34 there exists $\delta > 0$ such that for every $v \in X$

$$B_\delta(v) \subseteq V_x \quad \text{for some } x \in X \, . \tag{1.5.7}$$

Recall that this δ is called the Lebesgue number of the cover $\mathcal{L} = \{V_x\}_{x\in X}$; see Proposition 1.5.34 and Remark 1.5.35. Then, because of (1.5.6) and (1.5.7), $u \in B_\delta(v)$ implies $\rho(f(u), f(v)) < \varepsilon$. Hence, f is uniformly continuous. □

The next proposition is an easy consequence of the relevant definitions.

Proposition 1.5.40. (a) *Every metric space X is first countable.*
(b) *For a metric space X the notions of separability, second countability, and Lindelöf are all equivalent.*

Proof. (a) For every $x \in X$, let $\mathcal{B}(x) = \{B_r(x) : r \in \mathbb{Q}\}$. Then $\mathcal{B}$ is a countable local basis at $x \in X$. Therefore X is first countable.

(b) First we show that "separability" implies "second countability." Let $\{u_n\}_{n\geq 1}$ be dense in X. Then $\mathcal{B} = \{B_r(u_n) : r \in \mathbb{Q}, n \in \mathbb{N}\}$ is a countable basis of X, hence X is second countable. Theorem 1.2.27 says that "second countable" implies "Lindelöf." Finally we show that "Lindelöf" implies "separable." Consider the open cover $\{B_\varepsilon(x)\}_{x\in X}$ with $\varepsilon > 0$ of X. By the Lindelöf property there exists a countable subcover $\{B_\varepsilon(x_k)\}_{k\in\mathbb{N}}$. Let $A(\varepsilon) = \{x_k\}_{k\in\mathbb{N}}$. Then $D = \bigcup_{n\geq 1} A(1/n)$ is a countable dense subset of X. Therefore X is separable. □

Remark 1.5.41. In contrast to general topological spaces (see Proposition 1.2.22), for metric spaces, separability and second countability are equivalent notions.

Combining Proposition 1.5.40 with Theorem 1.4.63 we have the following result.

Theorem 1.5.42. *Let (X, d) be a metric space. Then the following assertions are equivalent:*
(a) *X is compact.*
(b) *X is countably compact.*
(c) *X is limit point compact.*
(c) *X is sequentially compact.*

Definition 1.5.43. A Hausdorff topological space (X, τ) is said to be **Polish** if it is separable and there exists a compatible metric d, that is $\tau = \tau_d$, for which X is complete.

Remark 1.5.44. In a Polish space the compatible metric is not a priori fixed. We know that it exists and generates the topology of X and that the space furnished with this metric is complete. There are many topological spaces that are Polish, but the corresponding complete metric is not particularly simple or natural. However, many constructions and facts depend only on the existence of a complete metric and not on the exact choice.

Proposition 1.5.45. *If X is a Polish space and $A \subseteq X$ is open or closed, then A is Polish.*

Proof. From Corollary 1.5.6 we know that A is separable. First suppose that A is open. We assume that $A \neq X$ and let d be the compatible metric on X for which X is complete.

Let

$$\hat{d}(x,u) = d(x,u) + \left| \frac{1}{d(x,A^c)} - \frac{1}{d(u,A^c)} \right| \quad \text{for all } x,u \in A\,. \tag{1.5.8}$$

It is easy to see that $\hat{d}$ is a metric on A. We show that $\hat{d}$ metrizes the subspace topology on A. From the triangle inequality we have

$$\left| d(x,A^c) - d(u,A^c) \right| \le d(x,u)\,,$$

which implies that $x \to d(x,A^c)$ is 1-Lipschitz, equivalently nonexpansive. Therefore, $u_n \xrightarrow{\hat{d}} u$ if and only if $u_n \xrightarrow{d} u$. Hence, $\hat{d}$ metrizes the subspace topology on A.

Suppose that $\{u_n\}_{n\ge1} \subseteq A$ is a $\hat{d}$-Cauchy sequence. Then, from (1.5.8) it is clear that $\{u_n\}_{n\ge1}$ is also a d-Cauchy sequence. Therefore, $u_n \xrightarrow{d} u \in X$. If $u \in A^c$, then $d(u_n, A^c) \to 0$ and so from (1.5.8) we have $\hat{d}(u_n,u_m) \to +\infty$ as $n,m \to +\infty$, a contradiction. Thus, $u \in A$ and so $u_n \xrightarrow{d} u$, which proves the completeness of $(A,\hat{d})$.

Now suppose that A is closed. Then $d_A = d|_{A\times A}$ is complete and so A is Polish. ☐

Proposition 1.5.46. *Countable products and countable intersections of Polish spaces are Polish spaces.*

Proof. For the products the result follows from Propositions 1.5.16, 1.5.17 and 1.5.18. For the intersections let

$$\Delta = \left\{ (u_n) \in \prod_{n\ge1} X_n :\ u_j = u_k \text{ for all } j,k \right\}\,.$$

Then Δ is closed, hence Polish; see Proposition 1.5.45. But Δ is homeomorphic to $\bigcap_{n\ge1} X_n$. ☐

The next result is known as “Alexandrov’s Theorem” and gives a characterization of Polish spaces.

Theorem 1.5.47 (Alexandrov’s Theorem). *If (X,τ) is a Polish space, then $A \subseteq X$ is Polish if and only if A is a G_δ-subset of X.*

Proof. $\Longrightarrow$: Let d be a compatible metric for X and d_0 a compatible complete metric for A. For each $n \in \mathbb{N}$, let V_n be the union of the open subsets U of X for which $U \cap A \ne \emptyset$ and d_0-diam$(U\cap A) < 1/n$, where d_0-diam denotes the diameter for the metric d_0. Since d and d_0 induce the same topology on A we have

$$A \subseteq \overline{A}^{\tau} \cap \left(\bigcap_{n\ge1} V_n \right)\,. \tag{1.5.9}$$

Let $u \in \overline{A}^{\tau} \cap (\bigcap_{n\ge1} V_n)$. Since $u \in \bigcap_{n\ge1} V_n$ we can find a sequence $\{U_n\}_{n\ge1}$ of neighborhoods of x such that

$$U_n \cap A \ne \emptyset \quad \text{and} \quad d_0\text{-}\operatorname{diam}(U_n \cap A) < \frac{1}{n}\,.$$

Evidently, by replacing U_n with a small neighborhood of u, we may assume that $\{U_n\}_{n\geq 1}$ is decreasing and d-diam $U_n \leq 1/n$. Since (A, d_0) is complete, from Theorem 1.5.15, we have that

$$\{u_0\} = \bigcap_{n\geq 1} \overline{U_n \cap A}^{\tau(A)} . \tag{1.5.10}$$

For every $n \in \mathbb{N}$ we have d-diam $\overline{U_n}^{\tau} \leq 1/n$ and $u, u_0 \in \overline{U_n}^{\tau}$. Hence, because of (1.5.10), $u = u_0$. Therefore, $\overline{A}^{\tau} \cap (\bigcap_{n\geq 1} V_n) \subseteq A$ and due to (1.5.9) it holds that $A = \overline{A}^{\tau} \cap (\bigcap_{n\geq 1} V_n)$. Invoking Proposition 1.5.8 for the closed $\overline{A}^{\tau}$, we conclude that A is a G_δ-subset of X.

$\Longleftarrow$: By hypothesis $A = \bigcap_{n\geq 1} U_n$ with $U_n \subseteq X$ open for all $n \in \mathbb{N}$. From Proposition 1.5.45 we know that each U_n is Polish and so Proposition 1.5.46 implies that $\bigcap_{n\geq 1} U_n = A$ is Polish. □

Remark 1.5.48. From the last theorem we recover the part of Proposition 1.5.45 concerning open sets.

Corollary 1.5.49. *The set of irrational numbers with the topology induced by* $\mathbb{R}$ *is Polish.*

Remark 1.5.50. We mention some more Polish spaces:
- Every locally compact, σ-compact metrizable space is Polish.
- Every locally compact and second countable Hausdorff space is Polish. This is a consequence of the so-called "Urysohn Metrization Theorem," which says that every regular, second countable space is metrizable.
- $\mathbb{N}^\infty$ is Polish (see Proposition 1.5.46) and in fact every Polish space is a continuous image of $\mathbb{N}^\infty$. More precisely every Polish space is a one-to-one continuous image of a closed subset of $\mathbb{N}^\infty$. On $\mathbb{N}^\infty$ we consider the tree metric defined by

$$t(\hat{p}, \hat{q}) = \begin{cases} 0 & \text{if } \hat{p} = \hat{q} \\ \frac{1}{k} & \text{if } \hat{p} \neq \hat{q} \text{ and } k = \min\{n \in \mathbb{N} \colon p_n \neq q_n\} \end{cases}$$

  for all $\hat{p} = (p_n), \hat{q} = (q_n) \in \mathbb{N}^\infty$. This is a complete metric on $\mathbb{N}^\infty$ compatible with the product topology.
- Every Polish space is a G_δ in some metrizable compactification.

Definition 1.5.51. A Hausdorff space X is said to be a **Souslin space** if there exist a Polish space Y and a continuous surjection $f \colon Y \to X$.

Remark 1.5.52. Equivalently we can say that the Hausdorff topological space (X, τ) is Souslin if and only if there is a topology $\tau_0 \supseteq \tau$ on X such that (X, τ_0) is homeomorphic to a quotient of a Polish space. A Souslin space is always separable but need not be metrizable. Anticipating some basic material from Chapter 3, we mention that an infinite dimensional separable Banach space with the weak topology is Souslin, but not metrizable. Similarly for the dual X^* of an infinite dimensional separable Banach space endowed with the w*-topology.

Definition 1.5.53. The Souslin subspaces of a Polish space are called **analytic sets**.

Souslin spaces have nice stability properties.

Proposition 1.5.54. (a) *Closed and open subsets of Souslin spaces are Souslin spaces.*
(b) *Countable products of Souslin spaces are Souslin.*
(c) *Countable intersections and countable unions of Souslin subspaces of a Hausdorff topological space V are Souslin.*

Proof. (a): Let X be a Souslin space. Then according to Definition 1.5.51 there exists a Polish space Y and a continuous surjection $f\colon Y \to X$. Let $E \subseteq X$ be a closed (resp. open) set. Then $f^{-1}(E) \subseteq Y$ is closed (resp. open) and so by Proposition 1.5.45 $f^{-1}(E)$ is Polish. Also $f|_{f^{-1}(E)}$ is continuous and surjective onto $f(f^{-1}(E)) = E$ since f is a surjection. Therefore, by Definition 1.5.51, E is Souslin.

(b): Let $\{X_n\}_{n\geq 1}$ be a family of Souslin spaces. For every $n \in \mathbb{N}$ there exists a Polish space Y_n and a continuous surjection $f_n\colon Y_n \to X_n$. Set $Y = \prod_{n\geq 1} Y_n, X = \prod_{n\geq 1} X_n$ and $\hat{f} = (f_n)_{n\geq 1}\colon X \to Y$ defined by $\hat{f}(\{y_n\}) = (f_n(y_n))_{n\geq 1}$. Then Y is Polish by Proposition 1.5.46 and $\hat{f}$ is a continuous surjection. So, X is a Souslin space.

(c): Let $\{X_n\}_{n\geq 1}$ be a family of Souslin subspaces of V and let $X = \prod_{n\geq 1} X_n$. We introduce $\hat{V} = V^{\mathbb{N}}$ and $\hat{\Delta}$ the diagonal of $\hat{V}$, that is, $\hat{\Delta} = \{\hat{u} = (u_n)_{n\geq 1}\colon u_n = u \text{ for all } n \in \mathbb{N}\}$. From Proposition 1.3.12 we know that $\hat{V}$ is Hausdorff and so Problem 1.1 implies that $\hat{\Delta} \subseteq \hat{V}$ is closed. Let $\hat{f}\colon V \to \hat{\Delta}$ be the canonical map of V onto $\hat{\Delta}$ defined by $\hat{f}(u) = (u, u, \dots, u, \dots)$. Then $\hat{f}(X) = \hat{\Delta} \cap (\prod_{n\geq 1} X_n)$ and $\hat{f}$ is a homeomorphism of X onto a closed subspaces of $\prod_{n\geq 1} X_n$. But by part (b) $\prod_{n\geq 1} X_n$ is Souslin, hence by part (a) $\hat{f}(X)$ is Souslin. Therefore X is Souslin.

Now we consider the union $\bigcup_{n\geq 1} X_n$. For every $n \in \mathbb{N}$ we can find a Polish space Y_n and a continuous surjection $f_n\colon Y_n \to X_n$. Let $\tilde{X}_n = \{n\} \times X_n$ and $\tilde{Y}_n = \{n\} \times Y_n$. Note that both are Polish spaces. Now we consider the map $\tilde{f}_n\colon \tilde{Y}_n \to \tilde{X}_n$ defined by $\tilde{f}_n(n, y) = (n, f_n(y))$ for all $n \in \mathbb{N}$ and for all $y \in Y_n$. Evidently $\tilde{f}_n$ is a continuous surjection. Let $\tilde{Y} = \bigcup_{n\geq 1} \tilde{Y}_n$ (this set is known as the free or disjoint union of the $Y_n's$ and sometimes it is denoted by $\sum_{n\geq 1} \tilde{Y}_n$) and similarly we set $\tilde{X} = \bigcup_{n\geq 1} \tilde{X}_n$. The function $\tilde{f}\colon \tilde{Y} \to \tilde{X}$ defined by $\tilde{f}|_{\tilde{Y}_n} = \tilde{f}_n$ for all $n \in \mathbb{N}$ is a continuous surjection. The space $\tilde{Y}$ is Polish; see Proposition 1.5.46. Let $h\colon \tilde{X} \to \bigcup_{n\geq 1} X_n$ be the canonical projection, that is, $h(n, u) = u$ for all $n \in \mathbb{N}$ and for all $u \in X_n$. This is a homeomorphism onto $\bigcup_{n\geq 1} X_n$. Then $g = h \circ \tilde{f}\colon \tilde{Y} \to \bigcup_{n\geq 1} X_n$ is a continuous surjection, hence $\bigcup_{n\geq 1} X_n$ is Souslin. $\square$

Directly from Definition 1.5.51, we have the following useful property of Souslin spaces. It shows that although Souslin spaces are not necessarily metrizable, they are sequentially determined.

Proposition 1.5.55. *If X is a Souslin space and $A \subseteq X$, then there exists a countable set $D \subseteq A$ such that D is sequentially dense in A.*

Proof. Let Y be a Polish space and $f\colon Y \to X$ a continuous surjection. Let $B = f^{-1}(A) \subseteq Y$. Then B is separable and so there exists a countable dense subset $D_0 \subseteq B$, that is,

$\overline{D}_0^Y \supseteq B$. Since f is surjective we know that $D = f(D_0) \subseteq A$ is countable and sequentially dense in A. □

Definition 1.5.56. A Hausdorff topological space X is said to be **strongly Lindelöf** if every open subset of X with the subspace topology is Lindelöf; see Definition 1.2.26(b).

Proposition 1.5.57. *Every Souslin space X is strongly Lindelöf.*

Proof. Let Y be a Polish space and $f: Y \to X$ a continuous surjection. Evidently Y is strongly Lindelöf; see Propositions 1.5.40(b) and 1.5.45. We can easily check that the continuous image of a strongly Lindelöf space is strongly Lindelöf. Hence X must be strongly Lindelöf. □

Definition 1.5.58. Let X, $\{Y_\alpha\}_{\alpha\in I}$ be sets and $f_\alpha: X \to Y_\alpha$ a family of functions. We say that the family $\{f_\alpha\}_{\alpha\in I}$ is **separating** (or **total**) if for every pair $(x, u) \in X \times X$ with $x \neq u$ we have $f_\alpha(x) \neq f_\alpha(u)$ for some $\alpha \in I$.

Lemma 1.5.59. *If X is a Souslin space, $\{Y_\alpha\}_{\alpha\in I}$ is a family of Hausdorff topological spaces and $f_\alpha: X \to Y_\alpha$ with $\alpha \in I$ is a separating family of continuous maps, then we can find a countable subset $D \subseteq I$ such that $\{f_\alpha\}_{\alpha\in D}$ remains separating.*

Proof. Replacing the $Y'_\alpha s$ by their free union (see the proof of Proposition 1.5.54(c)), we see that without any loss of generality we may assume that $Y_\alpha = Y$ for all $\alpha \in I$. Let $\Delta_X \subseteq X \times X$ and $\Delta_Y \subseteq Y \times Y$ be the diagonals. If $(x, u) \in \Delta_X^c$, then we can find $\alpha \in I$ such that $(f_\alpha(x), f_\alpha(u)) \in \Delta_Y^c$. So, the open sets $(f_\alpha, f_\alpha)^{-1}(\Delta_Y^c)$ with $\alpha \in I$ form an open cover of Δ_X^c. The space $X \times X$ is strongly Lindelöf; see Propositions 1.5.54(b) and 1.5.57. Therefore we can find a countable $D \subseteq I$ such that $\{(f_\alpha, f_\alpha)^{-1}(\Delta_Y)\}_{\alpha\in D}$ is a countable open cover of Δ_X^c. This means that $\{f_\alpha\}_{\alpha\in D}$ remains separating. □

Combining this lemma with Problem 1.41 we can state the following result concerning compact Souslin spaces.

Theorem 1.5.60. *Every compact Souslin space is metrizable, hence Polish.*

Remark 1.5.61. An improvement of this theorem can be found in Problem 1.42.

The Baire category notion gives a topological meaning to the notion of the size of a set. It is based on density. So, according to Baire, a subset A of a Hausdorff topological space X is considered to be very small (sparse) if there is no nonempty open set $U \subseteq X$ such that $A \cap U$ is dense in U, that is, $\overline{A}$ has an empty interior. Then large sets are those that are not countable unions of sparse sets.

Definition 1.5.62. Let X be a Hausdorff topological space and $A \subseteq X$.
(a) We say that A is **nowhere dense** if $\operatorname{int} \overline{A} = \emptyset$.
(b) We say that A is of **first category** if it is the countable union of nowhere dense sets.
(c) We say that A is of **second category** if it is not of first category.

Remark 1.5.63. Note that $\mathbb{Q}$ is of first category and at the same time dense in $\mathbb{R}$. The set $A \subseteq X$ is nowhere dense if and only if $\operatorname{int}(X \setminus A)$ is dense in X.

Definition 1.5.64. A Hausdorff topological space X is said to be a **Baire space** if the intersection of each countable family of dense, open sets in X is dense.

Proposition 1.5.65. *A Hausdorff topological space X is of second category in itself if and only if every countable family of dense open sets in X has nonempty intersection.*

Proof. $\Longrightarrow$: Let $\{U_n\}_{n\geq 1}$ be dense, open sets. Then $\{U_n^c\}_{n\geq 1} = \{X \setminus U_n\}_{n\geq 1}$ are nowhere dense, closed sets and so $\bigcup_{n\geq 1} U_n^c$ is of first category. Since by hypothesis X is of second category we have

$$X \setminus \left(\bigcup_{n\geq 1} U_n^c \right) = \bigcap_{n\geq 1} U_n \neq \emptyset .$$

$\Longleftarrow$: Arguing by contradiction, suppose that X is of first category. Then $X = \bigcup_{n\geq 1} C_n$ with C_n being nowhere dense and closed for each $n \in \mathbb{N}$. We have

$$X \setminus \left(\bigcup_{n\geq 1} C_n \right) = \bigcap_{n\geq 1} (X \setminus C_n) \neq \emptyset$$

since each $X \setminus C_n = U_n$ with $n \in \mathbb{N}$ is dense and open, a contradiction. This shows that X must be of second category. ☐

Proposition 1.5.66. *If X is a compact Hausdorff topological space and $A \subseteq X$ is a G_δ-set, then A is a Baire space.*

Proof. First we show that X is a Baire space. Let $\{U_n\}_{n\geq 1}$ be dense, open sets in X and let $V \subseteq X$ be a nonempty, open set. We have $U_1 \cap V \neq \emptyset$ and $U_1 \cap V$ is open. From Corollary 1.4.50 we know that X is normal, hence regular as well. So, we can find an open $W_1 \subseteq X$ such that $\overline{W_1} \subseteq U_1 \cap V$; see Proposition 1.2.8. Similarly, for $n \in \{2, 3, \ldots\}$ there exists open $W_n \subseteq X$ such that $\overline{W_n} \subseteq U_n \cap W_{n-1}$. Evidently $\{\overline{W_n}\}_{n\geq 1}$ is a decreasing sequence of compact sets, hence $\bigcap_{n\geq 1} \overline{W_n} \neq \emptyset$. But $\bigcap_{n\geq 1} \overline{W_n} \subseteq (\bigcap_{n\geq 1} U_n) \cap V$. So, every open set $V \subseteq X$ has a nonempty intersection with $\bigcap_{n\geq 1} U_n$ and this shows that $\bigcap_{n\geq 1} U_n$ is dense in X. Hence, X is a Baire space.

Without loss of generality we may assume that A is dense in X since we can always replace X by $\overline{A}$. Let $\{U_n\}_{n\geq 1}$ be dense, open subsets of A. Then $U_n = V_n \cap A$ with a dense and open $V_n \subseteq X$ for every $n \in \mathbb{N}$. Then

$$\bigcap_{n\geq 1} (V_n \cap A) = \left(\bigcap_{n\geq 1} V_n \right) \cap A .$$

From the first part of the proof we know that $\bigcap_{n\geq 1} V_n \subseteq X$ is dense. Therefore $\bigcap_{n\geq 1} U_n = \bigcap_{n\geq 1} (V_n \cap A)$ is dense in A. This proves that A is a Baire space. ☐

Corollary 1.5.67. *If X is a complete metric space and $X = \bigcup_{n\geq 1} C_n$ with closed $C_n \subseteq X$ for all $n \in \mathbb{N}$, then there exists a number $n_0 \in \mathbb{N}$ such that* $\operatorname{int} C_0 \neq \emptyset$.

Now Theorems 1.4.75 and 1.5.47 lead to the so-called “Baire Theorem.”

Theorem 1.5.68 (Baire Theorem). (a) *Every locally compact Hausdorff topological space is a Baire space.*
(b) *Every topologically complete Hausdorff space is a Baire space.*

We conclude this section with an important result known as “Stone’s Theorem.” For the proof we refer to Dugundji [91, p. 186].

Theorem 1.5.69 (Stone’s Theorem). *Every metrizable space is paracompact.*

1.6 Function Spaces

Let (X, τ_X) and (Y, τ_Y) be two Hausdorff topological spaces. By $C(X, Y)$ we denote the space of continuous functions $f: X \to Y$. In this section we topologize this space and study its properties.

Definition 1.6.1. Let $K \subseteq X$ be compact and $U \subseteq Y$ be open. We set

$$W(K, U) = \{f \in C(X, Y): f(K) \subseteq U\} .$$

The **compact-open topology** (or **c-topology**) on $C(X, Y)$ is the topology τ_ζ on $C(X, Y)$ having as subbasis the family

$$\{W(K, U): K \subseteq X \text{ is compact and } U \subseteq Y \text{ is open}\} .$$

Remark 1.6.2. A basic element for the τ_ζ-topology is given by

$$\bigcap_{n=1}^{m} W(K_n, U_n)$$

with compact $K_n \subseteq X$ and open $U_n \subseteq Y$ for all $n \in \{1, \ldots, m\}$. Note that $C(X, Y) \subseteq Y^X$. So, we can consider on $C(X, Y)$ the relative product topology that is the topology of pointwise convergence and is denoted by τ_p. Since $W(\{x\}, U) \in \tau_\zeta$ for all $x \in X$ and all open $U \subseteq Y$, it follows that

$$\tau_p \subseteq \tau_\zeta . \tag{1.6.1}$$

Note that we have

$$\bigcap_{n=1}^{m} W(K_n, U) = W\left(\bigcup_{n=1}^{m} K_n, U\right) , \quad \bigcap_{n=1}^{m} W(K, U_n) = W\left(K, \bigcap_{n=1}^{m} U_n\right) ,$$
$$\bigcap_{n=1}^{m} W(K_n, U_n) \subseteq W\left(\bigcup_{n=1}^{m} K_n, \bigcup_{n=1}^{m} U_n\right) , \quad \overline{W(K, U)}^{\tau_\zeta} \subseteq W(K, \overline{U}^{\tau_Y}) .$$

Proposition 1.6.3. *If (X, τ_X) and (Y, τ_Y) are Hausdorff topological spaces and the function space $C(X, Y)$ is endowed with the τ_ζ-topology, then the following hold:*

(a) *$C(X, Y)$ is Hausdorff;*
(b) *$C(X, Y)$ is regular if and only if Y is regular.*

Proof. (a) Let $f, g \in C(X, Y)$ such that $f \neq g$. We can find $x \in X$ such that $f(x) \neq g(x)$. Because Y is Hausdorff, we can find $U \in \mathcal{N}(f(x))$ and $V \in \mathcal{N}(g(x))$ such that $U \cap V = \emptyset$. Then

$$W(\{x\}, U) \in \tau_\zeta \text{ contains } f \,,$$
$$W(\{x\}, V) \in \tau_\zeta \text{ contains } g \,,$$
$$W(\{x\}, U) \cap W(\{x\}, V) = \emptyset \,.$$

This proves that $(C(X, Y), \tau_\zeta)$ is Hausdorff.

(b) $\Longrightarrow$: Evidently, $Y \subseteq C(X, Y)$ (the subspace of constant functions) and $\tau_\zeta(Y) = \tau_Y$. Then the regularity of Y follows from the fact that the property is hereditary; see Proposition 1.2.10.

$\Longleftarrow$: Let $f \in W(K, U)$. The set $f(K) \subseteq Y$ is compact. So, by Problem 1.52 we can find $V \in \tau_Y$ such that $f(K) \subseteq V \subseteq \overline{V} \subseteq U$. Then $f \in W(K, U) \subseteq \overline{W(K, U)}^{\tau_\zeta} \subseteq W(K, \overline{U}^{\tau_Y})$; see Remark 1.6.2. This proves that $(C(X, Y), \tau_\zeta)$ is regular. □

Remark 1.6.4. If Y is normal or first countable or second countable, then $(C(X, Y), \tau_\zeta)$ need not have the same properties.

Let (X, τ_X), (Y, τ_Y) and (Z, τ_Z) be three Hausdorff topological spaces. We can define the map $\eta\colon C(X, Y) \times C(Y, Z) \to C(X, Z)$ given by

$$\eta(f, g) = g \circ f \,. \tag{1.6.2}$$

On $C(X, Y)$, $C(Y, Z)$ and $C(X, Z)$ we consider the corresponding ζ-topologies.

Proposition 1.6.5. *The maps $f \to \eta(f, g)$ and $g \to \eta(f, g)$ are both continuous.*

Proof. We fix $f_1 \in C(X, Y)$ and prove the continuity of $g \to \eta(f_1, g)$ on $C(Y, Z)$. Let $W(K, U)$ be a subbasic neighborhood of $g \circ f_1$. Note that $g \circ f_1 \in W(K, U)$ if and only if $g \in W(f_1(K), U)$. But the set $f_1(K) \subseteq Y$ is compact. Hence, $W(f_1(K), U)$ is a subbasic neighborhood of g. Therefore, $\eta(f_1, W(f_1(K), U)) = W(K, U)$ and this proves the continuity of $g \to \eta(f_1, g)$.

Next we fix $g_1 \in C(Y, Z)$ and consider the map $f \to \eta(f, g_1)$ from $C(X, Y)$ into $C(X, Z)$. The proof of the continuity of this map is similar to the previous part. Note that in this case $g_1 \circ f \in W(K, U)$ if and only if $f \in W(K, g_1^{-1}(U))$ and $g^{-1}(U) \in \tau_Y$. □

To have joint continuity of the map η we need to strengthen the conditions on the space Y.

Proposition 1.6.6. *If (Y, τ_Y) is locally compact, then the map η is jointly continuous.*

Proof. Let $(f_1, g_1) \in C(X, Y) \times C(Y, Z)$ and let $W(K, U)$ be a subbasic neighborhood of (f_1, g_1). Note that $f_1(K) \subseteq g_1^{-1}(U)$, $f_1(K) \subseteq Y$ is compact and $g_1^{-1}(U) \subseteq Y$ is open. Since by hypothesis Y is locally compact, we can find relatively compact $V \in \tau_Y$ such that

$$f_1(K) \subseteq V \subseteq \overline{V} \subseteq g_1^{-1}(U) ;$$

see Proposition 1.4.66(c). Then we have

$$W(K, V) \subseteq \mathcal{N}(f_1) , \quad W(\overline{V}, U) \in \mathcal{N}(g_1) ,$$
$$\eta(W(K, V), W(\overline{V}, U)) \subseteq W(K, U) .$$

Hence, η is jointly continuous. □

Definition 1.6.7. The map $e\colon X \times C(X, Y) \to Y$ defined by $e(x, f) = f(x)$ is called the **evaluation map**. If we fix $x \in X$, the map $e_x\colon C(X, Y) \to Y$ defined by $e_x(f) = f(x)$ is called the **evaluation at** x **map**.

The next proposition establishes the continuity properties of these maps.

Proposition 1.6.8. (a) *If Y is locally compact, then $e\colon X \times C(X, Y) \to Y$ is continuous.*
(b) *For every $x \in X$, the map $e_x\colon C(X, Y) \to Y$ is continuous.*

Proof. Note that when Z is a singleton and $\eta\colon C(Z, X) \times C(X, Y) \to C(Z, Y)$ is the composition map (see (1.6.2)) then $\eta = e$. So, (a) follows from Proposition 1.6.6 while (b) follows from Proposition 1.6.5. □

We want to characterize the τ_ζ-compact subsets of $C(X, Y)$. The next definition introduces notions that are crucial in this direction.

Definition 1.6.9. Let (X, τ_X) be a Hausdorff topological space and (Y, d) be a metric space.

(a) A set $\mathcal{F} \subseteq C(X, Y)$ is said to be **equicontinuous** at x if for a given $\varepsilon > 0$ there exists $U \in \mathcal{N}(x)$ such that $d(f(u), f(x)) < \varepsilon$ for all $u \in U$ and for all $f \in \mathcal{F}$. We say that $\mathcal{F}$ is **equicontinuous** if it is equicontinuous at every $x \in X$.
(b) Given $f \in C(X, Y)$ with compact $K \subseteq X$ and $\varepsilon > 0$, we define

$$B_{K,\varepsilon}(f) = \{g \in C(X, Y)\colon \sup[d(g(x), f(x))\colon x \in K] < \varepsilon\} .$$

The sets $B_{K,\varepsilon}(f)$ form a basis for a topology τ_u on $C(X, Y)$ known as the **topology of uniform convergence on compacta**.

Remark 1.6.10. The τ_ζ-topology (see Definition 1.6.1) and the τ_p-topology (see Remark 1.6.2) on $C(X, Y)$ are defined without requiring that Y is a metric space. In contrast, the τ_u-topology (see Definition 1.6.9) explicitly requires that Y must be a metric space. Nevertheless, we can prove the following remarkable result.

Theorem 1.6.11. *If (X, τ_X) is a Hausdorff topological space and (Y, d) is a metric space, then $\tau_\zeta = \tau_u$.*

Proof. First we show that $\tau_\zeta \subseteq \tau_u$. To this end let $f \in W(K, U)$. Then $f(K) \subseteq Y$ is compact and $f(K) \subseteq U$.

Claim: There exists $\varepsilon > 0$ such that

$$f(K)_\varepsilon = \{y \in Y : d(y, f(K)) < \varepsilon\} \subseteq U .$$

Arguing by contradiction, suppose that the claim is not true. Then we can find $\{y_n\}_{n\geq 1} \subseteq Y \setminus U$ such that $d(y_n, f(K)) < 1/n$. Recall that $f(K) \subseteq Y$ is compact. So, for every $n \in \mathbb{N}$ there exists $v_n \in f(K)$ such that $d(y_n, v_n) = d(y_n, f(K)) < 1/n$ for all $n \in \mathbb{N}$. The compactness o $f(K)$ implies that by passing to a subsequence if necessary, we have $v_n \xrightarrow{d} v \in f(K)$ in Y. Since $d(y_n, v_n) < 1/n$ for all $n \in \mathbb{N}$, it follows that $y_n \xrightarrow{d} v \in (X \setminus U) \cap f(K)$, a contradiction, since $f(K) \subseteq U$. This proves that the claim is true.

The claim implies that $B_{K,\varepsilon}(f) \subseteq W(K, U)$, that is

$$\tau_\zeta \subseteq \tau_u . \tag{1.6.3}$$

Next we show that the opposite inclusion holds as well. Let $f \in C(X, Y)$ and let $B_{K,\varepsilon}(f) \subseteq W(K, U)$, see (1.6.3). For every $x \in X$ there exists $V_x \in \mathcal{N}(x)$ such that $f(\overline{V}_x) \subseteq U_x$ with $U_x \subseteq Y$ open and $\operatorname{diam} U_x < \varepsilon$. Since K is compact we find $x_1, \ldots, x_n \in K$ such that $K \subseteq \bigcup_{k=1}^n V_{x_k}$. Let $K_{x_k} = \overline{V}_{x_k} \cap K$ for $k \in \{1, \ldots, n\}$. Then $f \in \bigcap_{k=1}^n W(K_{x_k}, U_{x_k}) \subseteq B_{K,\varepsilon}(f)$ and so

$$\tau_u \subseteq \tau_\zeta . \tag{1.6.4}$$

From (1.6.3) and (1.6.4) it follows that $\tau_\zeta = \tau_u$. ◻

We know that $\tau_p \subseteq \tau_\zeta (= \tau_u$ if Y is a metric space); see (1.6.1) and Theorem 1.6.11. However, on equicontinuous sets, the two topologies coincide.

Proposition 1.6.12. *If (X, τ_X) is a Hausdorff topological space, (Y, d) is a metric space and $\mathcal{F} \subseteq C(X, Y)$ is equicontinuous, then $\tau_p(\mathcal{F}) = \tau_\zeta(\mathcal{F})$, that is, the two topologies restricted on $\mathcal{F}$ coincide.*

Proof. Evidently $\tau_p(\mathcal{F}) \subseteq \tau_\zeta(\mathcal{F})$. Moreover, Theorem 1.6.11 yields that $\tau_\zeta = \tau_u$. Therefore, it suffices to find a basic element B for the τ_p-topology such that

$$f \in B \cap \mathcal{F} \subseteq B_{K,\varepsilon}(f) \cap \mathcal{F} .$$

Let $\varepsilon_1, \varepsilon_2 > 0$ be such that $2\varepsilon_1 + \varepsilon_2 \leq \varepsilon$. Since $\mathcal{F}$ is equicontinuous and $K \subseteq X$ is compact, we find open sets $\{U_k\}_{k=1}^n$ in X such that $K \subseteq \bigcup_{k=1}^n U_k$ and for each $k \in \{1, \ldots, n\}$, each $x, u \in U_k$ and $f \in \mathcal{F}$, $d(f(x), f(u)) < \varepsilon_1$.

We choose $x_k \in U_k$ with $k \in \{1, \ldots, n\}$ and let

$$B = \{g \in C(X, Y) : d(g(x_k), f(x_k)) < \varepsilon_2 \text{ for all } k \in \{1, \ldots n\}\} .$$

Let $g \in B \cap \mathcal{F}$. Given $x \in K$, we find $k \in \{1, \ldots, n\}$ such that $x \in U_k$. Then we have

$$d(g(x), g(x_k)) \leq \varepsilon_1 , \quad d(g(x_k), f(x_k)) < \varepsilon_2 , \quad d(f(x_k), f(x)) \leq \varepsilon_1 ,$$

which implies, by the triangle inequality and the choice of $\varepsilon_1, \varepsilon_2 > 0$, that $d(g(x), f(x)) < \varepsilon$. Hence $g \in B_{K,\varepsilon}(f)$, thus $B \cap \mathcal{F} \subseteq B_{K,\varepsilon} \cap \mathcal{F}$. This proves that $\tau_p(\mathcal{F}) = \tau_\zeta(\mathcal{F})$. ◻

Proposition 1.6.13. *If (X, τ_X) is a Hausdorff topological space, (Y, d) is a metric space, and $\mathcal{F} \subseteq C(X, Y)$ is equicontinuous, then $\overline{\mathcal{F}}^{\tau_p}$ is equicontinuous as well.*

Proof. Let $x \in X$ and $\varepsilon > 0$. Since $\mathcal{F}$ is equicontinuous, there exists $U \in \mathcal{N}(x)$ such that $d(f(u), f(x)) < \varepsilon$ for all $u \in U$ and for all $f \in \mathcal{F}$.

Let $g \in \overline{\mathcal{F}}^{\tau_p}$. For $v \in U$ we introduce

$$V_v = \left\{h \in C(X, Y) \colon d(h(v), g(v)) < \frac{\varepsilon}{3}, d(h(x), g(x)) < \frac{\varepsilon}{3}\right\} \in \tau_p .$$

We have $V_v \cap \mathcal{F} \neq \emptyset$. Let $f \in V_v \cap \mathcal{F}$. We have

$$d(g(v), g(x)) \leq d(g(v), f(v)) + d(f(v), f(x)) + d(f(x), g(x)) \leq 3\frac{\varepsilon}{3} = \varepsilon .$$

Hence, $\overline{\mathcal{F}}^{\tau_p}$ is equicontinuous. □

The next theorem is the main result of this section and characterizes the τ_ζ-compact sets in $C(X, Y)$. The result is known as the “Arzela–Ascoli Theorem.”

Theorem 1.6.14 (Arzela–Ascoli Theorem). *If (X, τ_X) is a locally compact space, (Y, d) is a metric space, and $\mathcal{F} \subseteq C(X, Y)$, then $\overline{\mathcal{F}}^{\tau_\zeta}$ is τ_ζ-compact if and only if $\mathcal{F}$ is equicontinuous and for every $x \in X$, $\mathcal{F}(x) = \{f(x) \colon f \in \mathcal{F}\} \subseteq Y$ is relatively compact.*

Proof. $\Longrightarrow$: For every $x \in X$, there holds $\mathcal{F}(x) \subseteq \overline{\mathcal{F}}^{\tau_\zeta}(x) = e_x\left(\overline{\mathcal{F}}^{\tau_\zeta}\right)$ and Proposition 1.6.8(b) gives that $e_x\left(\overline{\mathcal{F}}^{\tau_\zeta}\right)$ is compact in Y. We need to show that $\overline{\mathcal{F}}^{\tau_\zeta}$ is equicontinuous. Let $x \in X$ and choose a compact set K such that $K \supseteq V \in \mathcal{N}(x)$. This is possible since X is supposed to be locally compact. Let $\mathcal{L}_\zeta = \{\hat{f} = f|_K \colon f \in \overline{\mathcal{F}}^{\tau_\zeta}\}$. It suffices to show that $\mathcal{L}_\zeta$ is equicontinuous. Let $r \colon C(X, Y) \to C(K, Y)$ be defined by $r(f) = f|_K$. Evidently $\mathcal{L}_\zeta = r\left(\overline{\mathcal{F}}^{\tau_\zeta}\right)$ and r is continuous when both $C(X, Y)$ and $C(K, Y)$ are endowed with their respective τ_ζ-topologies. Note that on $C(X, Y)$ the τ_ζ-topology coincides with the metric topology generated by the uniform metric $\hat{d}_K(f, g) = \max\{d(f(x), g(x)) \colon x \in K\}$. Hence $\mathcal{L}_\zeta$ is $\hat{d}_K$-totally bounded.

Let $\varepsilon > 0$ be given and choose $\varepsilon_1, \varepsilon_2 > 0$ such that $2\varepsilon_1 + \varepsilon_2 \leq \varepsilon$. We can find $x_1, \ldots, x_n \in K$ such that $\mathcal{L}_\zeta \subseteq \bigcup_{k=1}^n B_\varepsilon(\hat{f}_k)$. Since each $\hat{f}_K$ is continuous, we can find $U \in \mathcal{N}(x)$ such that

$$d\left(\hat{f}_k(u), \hat{f}_k(x)\right) < \varepsilon_2 \quad \text{for all } u \in U \text{ and for all } k \in \{1, \ldots, n\} . \tag{1.6.5}$$

Let $\hat{f} \in \mathcal{L}_\zeta$. Then $\hat{f} \in B_{\varepsilon_1}(\hat{f}_k)$ for some $k \in \{1, \ldots, n\}$. For every $u \in U$ we have

$$d\left(\hat{f}(u), \hat{f}_k(u)\right) < \varepsilon_1 , \quad d\left(\hat{f}_k(u), \hat{f}_k(x)\right) < \varepsilon_2 , \quad d\left(\hat{f}_k(x), \hat{f}(x)\right) < \varepsilon_1 ;$$

see (1.6.5). This gives $d\left(\hat{f}(u), \hat{f}(x)\right) < \varepsilon$ for all $u \in U$, which implies that $\mathcal{L}_\zeta$ is equicontinuous, and hence, so is $\mathcal{F}$.

$\Longleftarrow$: From Proposition 1.6.13 we know that $\overline{\mathcal{F}}^{\tau_p}$ is equicontinuous. Then Proposition 1.6.12 implies that $\overline{\mathcal{F}}^{\tau_p} = \overline{\mathcal{F}}^{\tau_\zeta}$. Recall that τ_p is the relative product topology on $C(X, Y) \subseteq Y^X$. Using Tychonoff’s Product Theorem (see Theorem 1.4.56), we have that

$\prod_{x\in X}\mathcal{F}(x)$ is compact in the product topology and so $\overline{\mathcal{F}}^{\tau_p}$ is compact. Therefore, $\overline{\mathcal{F}}^{\tau_\zeta}$ is compact. $\square$

A careful inspection of the second part of the proof above reveals that for that part of the result, the local compactness of X is not needed. So, we can state the following version of the Arzela–Ascoli Theorem.

Theorem 1.6.15. *If (X, τ_X) is a Hausdorff topological space, (Y, d) is a metric space, and $\mathcal{F} \subseteq C(X, Y)$ is a set with the following two properties:*
(a) *$\mathcal{F}$ is equicontinuous;*
(b) *for every $x \in X$, $\mathcal{F}(x) = \{f(x) : f \in \mathcal{F}\} \subseteq Y$ is relatively compact,*
then $\overline{\mathcal{F}}^{\tau_\zeta}$ is τ_ζ-compact and equicontinuous on X.

When $Y = \mathbb{R}^N$, exploiting the Heine–Borel Theorem, we can have the following particular version of the Arzela–Ascoli Theorem; see Theorem 1.6.14.

Theorem 1.6.16. *If (X, τ_X) is a compact topological space and $\mathcal{F} \subseteq C(X, \mathbb{R}^N)$, then $\mathcal{F}$ is compact for the supremum metric topology $\tau_{\hat{d}}$ if and only if $\mathcal{F}$ is equicontinuous, $\hat{d}$-closed, and bounded, that is, $|f(u)| \le M$ for all $u \in X$ and for some $M > 0$.*

Remark 1.6.17. If X is a compact space and (Y, d) is a metric space, then recall that the supremum metric $\hat{d}$ or d_∞ is defined by

$$\hat{d}(f, g) = d_\infty(f, g) = \max\{d(f(x), g(x)) : x \in X\} .$$

Evidently, $f_n \xrightarrow[d_\infty]{\hat{d}}$ if and only if $f_n \to f$ uniformly on X, that is, for given $\varepsilon > 0$, we can find $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $d(f_n(u), f(u)) \le \varepsilon$ for all $u \in X$ and for all $n \ge n_0$.

It is easy to see that uniform limits of continuous maps are again continuous maps. According to Theorem 1.6.11, the $\hat{d}$-metric topology depends only on the topology of Y and on the particular metric d. So, if d_1, d_2 are two compatible metrics on Y, then the corresponding sup-metrics $\hat{d}_1, \hat{d}_2$ are compatible as well. Hence we can view $C(X, Y)$ as a topological space without specifying the particular sup-metric and refer to the topology of uniform convergence on $C(X, Y)$.

Proposition 1.6.18. *If X is a compact metrizable space and Y is a separable metrizable space, then the space $C(X, Y)$ with the $\tau_\zeta = \tau_u$-topology is separable and metrizable.*

Proof. On account of Proposition 1.5.40(b) and Remark 1.6.17, it suffices to show that $C(X, Y)$ is second countable.

Let $D = \{x_n\}_{n\ge1} \subseteq X$ be a dense set and $\{U_n\}_{n\ge1}$ a countable basis for X. Let $\{\overline{B}_n\}_{n\ge1}$ be an enumeration of the countable set of all closed balls with center D and a rational radius. For $n, m \in \mathbb{N}$ let $W_{n,m} = W(\overline{B}_n, U_m)$.

We claim that $\{W_{n,m}\}_{n,m\ge1}$ is a countable subbasis for $C(X, Y)$. To this end, let $V \subseteq C(X, Y)$ be open and let $f \in V$. We choose $\delta > 0$ such that

$$B_{2\delta}(f) = \{g \in C(X, Y) : \hat{d}(g, f) < 2\delta\} \subseteq V .$$

Let d_Y be a compatible metric on Y and let $Y = \bigcup_{k\geq 1} V_k$ with $V_k \in \{U_n\}_{n\geq 1}$ and diam $V_k < \delta$. Moreover, let d_X be a compatible metric on X and write the open set $f^{-1}(V_k)$ as a union of d_X-balls with center $u_k \in X$, a rational radius, and closure in $f^{-1}(V_k)$. We have $X = \bigcup_{k\geq 1} f^{-1}(V_k)$ and the compactness of X implies that there exists a finite number of the balls $\overline{B}_n$ with $n \in \mathbb{N}$ such that $\bigcup_{i=1}^{k} \overline{B}_{n_i} = X$. For each i, choose m_i such that $\overline{B}_{n_i} \subseteq f^{-1}(U_{m_i})$. Let $g \in \bigcap_{i=1}^{k} W(\overline{B}_{n_i}, U_{m_i})$. If $x \in X$, we choose i such that $x \in \overline{B}_{n_i}$ and note that $f(x), g(x) \in U_{m_i}$. Since diam $U_{m_i} < \delta$, we have $d_Y(g(x), f(x)) < \delta$, which gives $\hat{d}(g, f) < \delta < 2\delta$. Hence $g \in B_{2\delta}(f) \subseteq V$. Therefore, $f \in \bigcap_{i=1}^{k} W(\overline{B}_{n_i}, W_{n_i}) \subseteq V$ and this proves the second countability of $C(X, Y)$. □

Remark 1.6.19. Combining Proposition 1.6.18 with Problem 1.21, we conclude that if Y is a Polish space, then so is $C(X, Y)$ equipped with the $\tau_\zeta = \tau_u$-topology.

1.7 Semicontinuous Functions – Miscellaneous Notions

In this section we examine semicontinuous extended real-valued functions and at the end we introduce some topological notions that arise in various parts of nonlinear analysis.

Semicontinuous $\mathbb{R}^*$-valued functions, where $\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$, provide a natural framework to study minimization or maximization problems with constraints. Here we will focus on lower semicontinuous $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$-valued functions. Of course with a minus sign all results can be reformulated for upper semicontinuous $\tilde{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$-valued functions.

So, let X be a set and let $\varphi\colon X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be a function. We introduce the following sets:

$$\operatorname{epi}\varphi = \{(u, \lambda) \in X \times \mathbb{R}\colon \varphi(u) \leq \lambda\} \text{ is the } \textbf{epigraph of } \varphi\,,$$
$$\varphi^\lambda = \{u \in X\colon \varphi(u) \leq \lambda\} \text{ with } \lambda \in \mathbb{R} \text{ is the } \lambda\textbf{-sublevel set of } \varphi\,,$$
$$\operatorname{dom}\varphi = \{u \in X\colon \varphi(u) < +\infty\} \text{ is the } \textbf{effective domain of } \varphi\,.$$

To avoid trivial situations, we will always consider functions with dom $\varphi \neq \emptyset$. In the optimization literature such functions are called **proper**. However, in nonlinear analysis, this name is reserved for maps that have the property where the inverse image of a compact set is compact.

Note that if $\{\varphi_\alpha\}_{\alpha\in I}$ is a family of $\overline{R}$-valued functions then

$$\operatorname{epi}\left(\sup_{\alpha\in I}\varphi_\alpha\right) = \bigcap_{\alpha\in I}\operatorname{epi}\varphi_\alpha\,, \tag{1.7.1}$$

$$\operatorname{epi}\left(\inf_{\alpha\in I}\varphi_\alpha\right) = \bigcup_{\alpha\in I}\operatorname{epi}\varphi_\alpha\,. \tag{1.7.2}$$

Definition 1.7.1. Let (X, τ) be a Hausdorff topological space and $\varphi\colon X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. We say that φ is τ**-lower semicontinuous at** $x \in X$ if for every $\lambda < \varphi(x)$ there exists

$U_\lambda \in \mathcal{N}(x)$ such that $\lambda < f(u)$ for all $u \in U_\lambda$. We say that φ is **τ-lower semicontinuous** if it is τ-lower semicontinuous at every $x \in X$.

Proposition 1.7.2. *If (X, τ) is a Hausdorff topological space and $\varphi\colon X \to \overline{\mathbb{R}}$ a function, then the following statements are equivalent:*
(a) *φ is τ-lower semicontinuous;*
(b) $\operatorname{epi}\varphi \subseteq X \times \mathbb{R}$ *is closed (we consider the product topology on* $X \times \mathbb{R}$*);*
(c) *for every $\lambda \in \mathbb{R}$, $\varphi^\lambda \subseteq X$ is closed;*
(d) $\varphi(x) \le \liminf_{u\to x} \varphi(u) = \sup_{U\in\mathcal{N}(x)} \inf_{u\in U} \varphi(u)$ *for all $x \in X$.*

Proof. (a) $\Longrightarrow$ (b): Let $(u, \mu) \notin \operatorname{epi}\varphi$. Then $\mu < \varphi(u)$. Let $\eta \in (\mu, \varphi(u))$. Then by Definition 1.7.1, there exists $U_\eta \in \mathcal{N}(u)$ such that $\mu < \eta < \varphi(v)$ for all $v \in U_\eta$. Then

$$\big(U_\eta \times (-\infty, \eta)\big) \cap \operatorname{epi}\varphi = \emptyset\,.$$

Since $U_\eta \times (-\infty, \eta)$ is a neighborhood of (u, λ) in $X \times \mathbb{R}$, we conclude that $(X \times \mathbb{R}) \setminus \operatorname{epi}\varphi$ is open, hence $\operatorname{epi}\varphi$ is closed in $X \times \mathbb{R}$ with the product topology.

(b) $\Longrightarrow$ (c): Note that $\varphi^\lambda \times \{\lambda\} = \operatorname{epi}\varphi \cap (X \times \{\lambda\})$. Therefore $\varphi^\lambda \times \{\lambda\}$ is closed in $X \times \mathbb{R}$. But the map $u \to (u, \lambda)$ is a homeomorphism from X onto $X \times \{\lambda\}$. Therefore φ^λ is closed.

(c) $\Longrightarrow$ (d): Let $\lambda < \varphi(x)$. Since by hypothesis $X \setminus \varphi^\lambda$ is open, we can find $U \in \mathcal{N}(x)$ such that $U \subseteq \big(X \setminus \varphi^\lambda\big)$. So, we have $\lambda \le \inf_U \varphi$, which implies $\lambda \le \sup_{U\in\mathcal{N}(x)} \inf_{u\in U} \varphi(u) = \liminf_{u\to x} \varphi(u)$. Since $\lambda < \varphi(x)$ is arbitrary we let $\lambda \nearrow \varphi(x)$ to conclude that $\varphi(x) \le \liminf_{u\to x} \varphi(u)$.

(d) $\Longrightarrow$ (a): Let $\lambda < \varphi(x)$. By hypothesis $\lambda < \sup_{U\in\mathcal{N}(x)} \inf_{u\in U} \varphi(u)$ and thus $\lambda < \inf_{u\in U_0} \varphi(u)$ for some $U_0 \in \mathcal{N}(x)$. Hence, φ is τ-lower semicontinuous at any $x \in X$. ☐

Remark 1.7.3. If $\varphi\colon X \to \tilde{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, then instead we use the **hypograph** $\operatorname{hyp}\varphi = \{(u, \lambda) \in X \times \mathbb{R}\colon \lambda \le \varphi(u)\}$ and the **λ-superlevel set** $\varphi_\lambda = \{u \in X\colon \varphi(u) \ge \lambda\}$. We have that φ is upper semicontinuous if and only if $\operatorname{hyp}\varphi$ is closed if and only if for $\lambda \in \mathbb{R}$, φ_λ is closed if and only if $\varphi(x) \ge \limsup_{u\to x} \varphi(u) = \inf_{U\in\mathcal{N}(x)} \sup_{u\in U} \varphi(u)$ for all $x \in X$.

Proposition 1.7.2 leads to some useful stability properties for lower semicontinuous functions.

Proposition 1.7.4. *If (X, τ) is a Hausdorff topological space and $\varphi_\alpha\colon X \to \overline{\mathbb{R}}$ with $\alpha \in I$, is a family of τ-lower semicontinuous functions, then the following hold:*
(a) $\sup_{\alpha\in I} \varphi_\alpha$ *is τ-lower semicontinuous;*
(b) *if I is finite, then* $\inf_{\alpha\in I} \varphi_\alpha$ *is τ-lower semicontinuous.*

Proof. (a) This follows from (1.7.1) and Proposition 1.7.2.

(b) Since I is finite and the finite union of closed sets is closed, the result follows from (1.7.2) and Proposition 1.7.2. ☐

Similarly, using Proposition 1.7.2, we have the following result.

Proposition 1.7.5. *If (X, τ) is a Hausdorff topological space and $\varphi, \psi\colon X \to \overline{\mathbb{R}}$ are τ-lower semicontinuous functions, then $\varphi + \psi$ is τ-lower semicontinuous.*

On metric spaces semicontinuous functions can be realized as monotone limits of Lipschitz functions.

Proposition 1.7.6. *If (X, d) is a metric space and $\varphi\colon X \to \overline{\mathbb{R}}$ is bounded from below, then φ is lower semicontinuous if and only if there exists an increasing sequence of Lipschitz continuous bounded functions $\hat{\varphi}_n\colon X \to \mathbb{R}$ such that $\hat{\varphi}_n(u) \nearrow \varphi(u)$ for all $u \in X$.*

Proof. $\Longrightarrow$: For every $n \in \mathbb{N}$ let $\varphi_n\colon X \to \mathbb{R}$ be defined by

$$\varphi_n(u) = \inf[\varphi(x) + nd(x, u)\colon x \in X] . \tag{1.7.3}$$

Clearly $\{\varphi_n\}_{n\geq 1}$ is increasing and $\varphi_n \leq \varphi$ for every $n \in \mathbb{N}$. Moreover, for every $v \in X$ we have

$$\varphi_n(u) \leq \varphi(x) + nd(x, u) \leq \varphi(x) + nd(x, v) + nd(v, u) \quad \text{for all } x \in X .$$

This gives $\varphi_n(u) \leq \varphi_n(v) + nd(v, u)$, hence $|\varphi_n(u) - \varphi_n(v)| \leq nd(v, u)$. Thus each φ_n is Lipschitz.

We have $\varphi_n(u) \nearrow \tilde{\varphi}(u) \leq \varphi(u)$ for all $u \in X$. Given $\varepsilon > 0$, from (1.7.3), we see that there exists $x_n \in X$ such that

$$\varphi(x_n) + nd(x_n, u) \leq \varphi_n(u) + \varepsilon . \tag{1.7.4}$$

Let $\eta \leq \varphi(x)$ for all $x \in X$. So, from (1.7.4), we have

$$d(x_n, u) \leq \frac{1}{n}[\varphi_n(u) + \varepsilon - \eta] . \tag{1.7.5}$$

Hence, if $u \in \operatorname{dom}\varphi$, then $d(x_n, u) \leq 1/n[\varphi(u) + \varepsilon - \eta]$, which shows that

$$x_n \xrightarrow{d} u . \tag{1.7.6}$$

Hence if we pass to the limit as $n \to \infty$ in (1.7.4) and use (1.7.6), then $\varphi(u) \leq \tilde{\varphi}(u) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we let $\varepsilon \searrow 0$ and obtain $\varphi(u) \leq \tilde{\varphi}(u)$, which implies $\varphi(u) = \tilde{\varphi}(u)$ for all $u \in \operatorname{dom}\varphi$.

If $u \notin \operatorname{dom}\varphi$, then we claim that $\tilde{\varphi}(u) = +\infty$. Indeed if $\tilde{\varphi}(u) \in \mathbb{R}$, then from (1.7.5) we have

$$d(x_n, u) \leq \frac{1}{n}\left[\tilde{\varphi}(u) + \varepsilon - \eta\right] .$$

Hence, $x_n \xrightarrow{d} u$. So, as above we obtain $+\infty = \varphi(u) \leq \tilde{\varphi}(u) < +\infty$, a contradiction. Thus $\varphi_n(u) \nearrow +\infty$ for all $u \notin \operatorname{dom}\varphi$. Finally let $\hat{\varphi}_n = \min\{\varphi_n, n\}$. Then $\hat{\varphi}_n$ is bounded as well. □

Remark 1.7.7. If $\varphi\colon X \to \tilde{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ is upper semicontinuous and bounded above, then we can find a decreasing sequence of Lipschitz continuous bounded functions $\hat{\varphi}_n\colon X \to \mathbb{R}$ such that $\hat{\varphi}_n(u) \to \varphi(u)$ for all $u \in X$ as $n \to \infty$.

From Proposition 1.7.6 and Remark 1.7.7, we infer the following useful result.

Corollary 1.7.8. *If (X, d) is a metric space and $\varphi \in C_b(X, \mathbb{R})$, then there exist two sequences of Lipschitz continuous bounded functions $\xi_n, \eta_n : X \to \mathbb{R}$ such that*
(a) *$\{\xi_n\}_{n\geq 1}$ is increasing and $\xi_n(u) \nearrow \varphi(u)$ for all $u \in X$;*
(b) *$\{\eta_n\}_{n\geq 1}$ is decreasing and $\eta_n(u) \searrow \varphi(u)$ for all $u \in X$.*

In general pointwise convergence of functions does not imply uniform convergence. However, with additional hypotheses we can have this. The result is known as "Dini's Theorem."

Theorem 1.7.9 (Dini's Theorem). *If (X, τ) is a countably compact Hausdorff topological space, $\varphi_n : X \to \mathbb{R}$ with $n \in \mathbb{N}$ is an increasing (resp. decreasing) sequence of lower (resp. upper) semicontinuous functions and $\varphi_n(u) \to \varphi(u)$ for all $u \in X$ with $\varphi : X \to \mathbb{R}$ upper (resp. lower) semicontinuous, then φ is continuous and $\varphi_n \to \varphi$ uniformly, that is $\hat{d}(\varphi_n, \varphi) = \sup_{x\in X} |\varphi_n(x) - \varphi(x)| \to 0$ as $n \to \infty$.*

Proof. We do the case of a lower semicontinuous sequence. The other case is obtained by multiplying with -1. From Proposition 1.7.4(a), we have that φ is lower semicontinuous as well, hence continuous. Then, for all $n \in \mathbb{N}$, $\varphi_n - \varphi \leq 0$ and it is lower semicontinuous. Given $\varepsilon > 0$, let $U_n = \{u \in X : (\varphi_n - \varphi)(u) > -\varepsilon\}$. Then $\{U_n\}_{n\geq 1}$ is an open cover of X and so by countable compactness we can find a finite subcover; see Definition 1.4.57(a). Since $\{U_n\}_{n\geq 1}$ are increasing, then for some $n \in \mathbb{N}$, $U_n = X$. Hence $-\varepsilon < (\varphi_m - \varphi)(u) \leq 0$ for all $m \geq n$. Therefore, $\varphi_n \to \varphi$ uniformly on X. □

Remark 1.7.10. The hypotheses in Theorem 1.7.9 can not be relaxed. Let $\varphi_n(x) = x^n$ for all $x \in [0, 1)$. Then $\varphi_n \searrow 0$ but the convergence is not uniform. The domain $[0, 1)$ is not compact. Moreover, if $X = [0, 1]$, then $\varphi_n(x) = x^n \to \chi_{\{1\}}(x)$ and again the convergence is not uniform since $\chi_{\{1\}}$ is not lower semicontinuous. Note that the characteristic function

$$\chi_C(x) = \begin{cases} 1 & \text{if } x \in C\,, \\ 0 & \text{if } x \notin C \end{cases}$$

of a closed set C is only upper semicontinuous.

Next we introduce some topological notions that are used often in problems of nonlinear analysis.

Definition 1.7.11. Let (X, τ) be a Hausdorff topological space and $A \subseteq X$. We say that A is a **retract** of X if there is a continuous map $r : X \to A$ such that $r|_A = \mathrm{id}\,|_A$. The map $r : X \to A$ is called a **retraction**.

Remark 1.7.12. Equivalently we can say that $A \subseteq X$ is a retract of X if $\mathrm{id}\,|_A$ is continuously extendable to X. The concept of retracts is a topological notion, that is, if $h : X \to Y$ is a homeomorphism and $A \subseteq X$ is a retract of X, then $h(A)$ is a retract of Y.

Example 1.7.13. (a) X and for $u \in X$, the singletons $\{u\}$ are retracts of X.

(b) If $\overline{B}_1^n = \{u \in \mathbb{R}^n : |u| \le 1\}$ and $S^{n-1} = \{u \in \mathbb{R}^n : |u| = 1\}$, then $\overline{B}_1^n$ is a retract of $\mathbb{R}^n$ with a retraction given by

$$r(u) = \begin{cases} \frac{u}{|u|} & \text{if } |u| \ge 1\,, \\ u & \text{if } |u| < 1\,, \end{cases}$$

while S^{n-1} is a retract of $\mathbb{R}^n \setminus \{0\}$ with a retraction given by $r(u) = u/|u|$ for all $u \in \mathbb{R}^n \setminus \{0\}$.

(c) Every nonempty closed subset of the Polish space $\mathbb{N}^\infty$ is a retract of $\mathbb{N}^\infty$.

Proposition 1.7.14. *If (X, τ) is a Hausdorff topological space and A is a retract of X, then A is closed.*

Proof. Arguing by contradiction, suppose that A is not closed and let $x \in \overline{A} \setminus A$. Then, for a retraction r, we have $r(x) \neq x$ and so we can find $U \in \mathcal{N}(x)$, $V \in \mathcal{N}(r(x))$ such that $U \cap V = \emptyset$ since X is assumed to be Hausdorff. Because of the continuity of r, there holds $r(U) \subseteq V$. Let $u \in A \cap U$, recall $x \in \overline{A}$, then $r(u) = u \in V$, a contradiction. □

Proposition 1.7.15. *If X is a Hausdorff topological space and $A \subseteq X$, then A is a retract of X if and only if for every Hausdorff topological space Y every continuous map $f : A \to Y$ is continuously extendable on all of X.*

Proof. $\Longrightarrow$: Let $r : X \to A$ be a retraction. Then $f \circ r : X \to Y$ is a continuous extension of f.

$\Longleftarrow$: Let $Y = A$. Then, according to Remark 1.7.12, A is a retract of X. □

Definition 1.7.16. Let X, Y be two Hausdorff topological spaces and $f, g : X \to Y$ two continuous maps. A **homotopy** from f to g is a continuous map $h : [0, 1] \times X \to Y$ such that $h(0, \cdot) = f(\cdot)$ and $h(1, \cdot) = g(\cdot)$. Then we say that f and g are **homotopic** and write $f \simeq g$ (or $f \simeq g$ (h) if we need to emphasize the homotopy).

Remark 1.7.17. We can think of the homotopy as a time dependent deformation, with the parameter $t \in [0, 1]$ being the time, of f into g as time moves from 0 to 1. This deformation is continuous. So there are no breaks or jumps.

Proposition 1.7.18. *$\simeq$ is an equivalence relation on $C(X, Y)$.*

Proof. First, we see that $f \simeq f$ via the constant homotopy $h(t, \cdot) = f(\cdot)$ for all $t \in [0, 1]$. Now let $f, g \in C(X, Y)$ and suppose that $f \simeq g$. Denote by $h : [0, 1] \times X \to Y$ the corresponding homotopy. Then $\tilde{h}(t, x) = h(1 - t, x)$ for all $t \in [0, 1]$ and for all $x \in X$ is a homotopy from g to f. Therefore $g \simeq f$. Finally if $f \simeq g$ (h_1) and $g \simeq k(h_2)$, then

$$h(t, x) = \begin{cases} h_1(2t, x) & \text{if } x \in [0, \frac{1}{2}]\,, \\ h_2(2t - 1, x) & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

for all $t \in [0, 1]$ and for all $x \in X$ is a homotopy from f to k; see Proposition 1.1.37. Hence, $f \simeq k$. □

Definition 1.7.19. Let X, Y be two Hausdorff topological spaces.
(a) If $f \in C(X, Y)$ is homotopic to a constant map, then we say that f is **nullhomotopic** and we write that $f \simeq 0$.
(b) We say that the space X is **contractible** if id_X is nullhomotopic.
(c) If $\varphi \in C(X, Y)$ and $\psi \in C(Y, X)$, then we say that ψ is a **homotopy inverse** of φ if $\psi \circ \varphi \simeq \mathrm{id}_X$ and $\varphi \circ \psi \simeq \mathrm{id}_Y$. If φ has a homotopy inverse, then φ is said to be a **homotopy equivalence**. In this case we say that X and Y are **homotopy equivalent** (or of the same **homotopy type**).

Remark 1.7.20. It is easy to check by applying Proposition 1.7.18 that homotopy equivalence is an equivalence relation. Note that every convex set in $\mathbb{R}^N$ is contractible and, more generally, every star-shaped set in $\mathbb{R}^N$ is contractible. Recall that a set $A \subseteq \mathbb{R}^N$ is **star-shaped**, if there exists $u_0 \in A$ such that for every $u \in A$, the line segment $[u_0, u] = \{(1-t)u_0 + tu\colon 0 \le t \le 1\}$ is contained in A. In general, a contractible space is one that can be continuously shrunk to a point. Indeed, according to Definition 1.7.19(b), there exists a continuous map $h\colon [0,1] \times X \to X$ such that $h(0, x) = x$ for all $x \in X$ and $h(1, x) = x_0$ for all $x \in X$ with $x_0 \in X$.

Definition 1.7.21. Let X be a Hausdorff topological space.
(a) A continuous map $h\colon [0,1] \times X \to X$ is a **deformation** of X if $h(0, \cdot) = \mathrm{id}_X$. Moreover, if $h(1, X) \subseteq A \subseteq X$, then we say that h is a deformation of X onto A.
(b) A closed set $A \subseteq X$ is a **(resp. strong) deformation retract** of X if there exists a deformation $h\colon [0,1] \times X \to X$ of X onto A such that $h(1, \cdot)\big|_A = \mathrm{id}_A$ (resp. such that $h(t, \cdot)\big|_A = \mathrm{id}_A$ for all $t \in [0,1]$). The deformation h is called a (resp. strong) deformation retraction.

Remark 1.7.22. Note that $A \subseteq X$ is a deformation retract if and only if there exists a retraction $r\colon X \to A$ (see Definition 1.7.11), such that $i_A \circ r \simeq \mathrm{id}_X$; see Definition 1.7.16. Then, since $r \circ i_A = \mathrm{id}_A$, we infer that the inclusion map $i_A\colon A \to X$ is a homotopy equivalence.

Example 1.7.23. From Example 1.7.13(b), we know that S^n is a retract of $\mathbb{R}^{n+1} \setminus \{0\}$. In fact it is a strong deformation retract. Indeed, consider the deformation $h\colon [0,1] \times (\mathbb{R}^{n+1} \setminus \{0\}) \to \mathbb{R}^{n+1}$ defined by

$$h(t, x) = (1-t)x + t\frac{x}{|x|} \quad \text{for all } t \in [0,1] \text{ and for all } x \in \mathbb{R}^{n+1} \setminus \{0\}\,.$$

Directly from the previous definitions we have the following result.

Proposition 1.7.24. *If X is a Hausdorff topological space, then the following statements are equivalent:*
(a) *X is contractible.*
(b) *X is homotopy equivalent to a singleton.*
(c) *Any point of X is a deformation retract of X.*

Proposition 1.7.25. *If Y is a Hausdorff topological space, then $f \in C(S^n, Y)$ is nullhomotopic if and only if there exists a $\hat{f} \in C(\overline{B}_1^n, Y)$ such that $\hat{f}|_{S^n} = f$, that is, $\hat{f}$ is a continuous extension of f on $\overline{B}_1^n$.*

Proof. $\Longrightarrow$: Since $0 \simeq f$, there exists a homotopy $h\colon [0,1]\times S^n \to Y$ such that $h(0,\cdot) = u_0$ and $h(1,\cdot) = f$. Let

$$\hat{f}(x) = \begin{cases} u_0 & \text{if } 0 \le |x| \le \frac{1}{2}\,, \\ h\left(2|x| - 1, \frac{x}{|x|}\right) & \text{if } \frac{1}{2} \le |x| \le 1\,. \end{cases}$$

Then $\hat{f} \in C(\overline{B}_1^n, Y)$ and $\hat{f}|_{S^n} = f$.

$\Longleftarrow$: Let $h(t,x) = \hat{f}(tx)$ for all $t \in [0,1]$ and for all $x \in \overline{B}_1^n$. Then, using this homotopy, we see that $0 \simeq f$. ☐

The next notion is related to the Tietze Extension Theorem; see Theorem 1.2.44.

Definition 1.7.26. A Hausdorff topological space X is said to be an **absolute retract** (AR for short) if the following are true:
(a) X is metrizable;
(b) for any metrizable space Y and any closed set $A \subseteq Y$ each $f \in C(A, X)$ can be extended to a $\hat{f} \in C(Y, X)$, that is, $\hat{f}|_A = f$.

Remark 1.7.27. So an AR can replace $\mathbb{R}$ in the Tietze Extension Theorem, see Theorem 1.2.44, for metric spaces.

Proposition 1.7.28. *If X is an* AR *and C is a retract of X, then C is an* AR.

Proof. Let Y be a metrizable space, $A \subseteq Y$ a closed set, and $f \in C(A, C)$. Let $r\colon X \to C$ be a retraction. Since X is an AR, there exists $\hat{f} \in C(Y, X)$ such that $\hat{f}|_A = f$. Then $\hat{f}_0 = r \circ \hat{f} \in C(Y, C)$ is the desired extension of f. ☐

Now we will identify some useful spaces that are AR. The first result is known as "Dugundji's Extension Theorem."

Theorem 1.7.29 (Dugundji's Extension Theorem). *If X is a metrizable space, $A \subseteq X$ is closed, Y is a locally convex space, and $f \in C(A, Y)$, then there exists $\hat{f} \in C(X, Y)$ such that $\hat{f}|_A = f$ and $\hat{f}(X) \subseteq \operatorname{conv} f(A)$.*

Proof. Let d be a compatible metric on X. For $x \in X$ and $r > 0$, let $B(x, r) = \{u \in X\colon d(u, x) < r\}$. We consider the family $\{B(x, 1/2d(x, A))\colon x \in X \setminus A\}$. This is an open cover of $X \setminus A$. Since $X \setminus A$ is paracompact (see Theorem 1.5.69), there exists a locally finite refinement $\{U_\alpha\}_{\alpha\in I}$. For U_α choose $B(x_\alpha, 1/2d(x_\alpha, A))$ such that

$$U_\alpha \subseteq B\left(x_\alpha, \frac{1}{2}d(x_\alpha, A)\right)\,; \tag{1.7.7}$$

see Definition 1.4.79(a). We choose $u_\alpha \in A$ such that

$$d(x_\alpha, u_\alpha) \le 2d(x_\alpha, A)\,. \tag{1.7.8}$$

We have

$$d(x_\alpha, A) \le 2d(x, A) \quad \text{for all } x \in U_\alpha\,. \tag{1.7.9}$$

To see (1.7.9) note that for all $x \in U_\alpha$

$$d(x_\alpha, A) \le d(x_\alpha, x) + d(x, A) \le \frac{1}{2} d(x_\alpha, A) + d(x, A)\,;$$

see (1.7.7). Hence, (1.7.9) holds.

Moreover we have

$$d(u, u_\alpha) \le 6d(u, x) \quad \text{for all } u \in A \text{ and all } x \in U_\alpha\,. \tag{1.7.10}$$

Again, to see (1.7.10), note that, because of (1.7.7) and (1.7.8), for all $u \in A$ and for all $x \in U_\alpha$,

$$\begin{aligned} d(u, u_\alpha) &\le d(u, x) + d(x, x_\alpha) + d(x_\alpha, u_\alpha) \\ &\le d(u, x) + \frac{1}{2} d(x_\alpha, A) + 2d(x_\alpha, A) \\ &\le d(u, x) + d(x, A) + 4d(x, A) \\ &\le 6d(u, x)\,. \end{aligned}$$

Thus, (1.7.10) holds.

Invoking Theorem 1.4.86, there exists a partition of unity $\{\xi_\alpha\}_{\alpha\in I}$ subordinated to the cover $\{U_\alpha\}_{\alpha\in I}$. We define

$$\hat{f}(u) = \begin{cases} f(u) & \text{if } u \in A\,, \\ \sum_{\alpha\in I} \xi_\alpha(u) f(u_\alpha) & \text{if } u \in X \setminus A\,. \end{cases} \tag{1.7.11}$$

Clearly, $\hat{f}\big|_A = f$ and $\hat{f}$ is continuous on the open set $X \setminus A$. We need to show the continuity of f at the points of A.

Let $u \in A$ and $V \in \mathcal{N}(f(u))$. Since Y is locally convex and f is continuous at u, we can find a convex set C and a $\delta > 0$ such that

$$f\left(A \cap B_{\frac{\delta}{6}}(u)\right) \subseteq C \subseteq V\,. \tag{1.7.12}$$

Let x be any point of $B_{\delta/6}(u) \setminus A$. Since the cover $\{U_\alpha\}_{\alpha\in I}$ is locally finite, it belongs to finitely many sets $U_{\alpha_1}, \dots, U_{\alpha_n}$. Then $d(x, u) < \delta/6$ and since $x \in U_\alpha$ we have $d(u, u_{\alpha_i}) < \delta$ for all $i \in \{1, \dots, n\}$; see (1.7.10). This implies that $u_{\alpha_i} \in A \cap B_\delta(u)$ for all $i \in \{1, \dots, n\}$. Because of (1.7.11) and since C is convex it follows that $\hat{f}(u) \in C$. Therefore, due to (1.7.12), $\hat{f}(B_{\delta/6}(u)) \subseteq V$. Hence $\hat{f}\big|_A$ is continuous. □

Corollary 1.7.30. *If C is a convex subset of a locally convex space X and C is metrizable, then C is an* AR.

Next we show that in an infinite dimensional normed space X, the unit sphere $\partial B_1 = \{u \in X\colon \|u\| = 1\}$ is an AR. To do this we will need the following remarkable result due to Klee [176].

Theorem 1.7.31. *If X is an infinite dimensional normed space and $K \subseteq X$ is compact, then $X \setminus C$ and X are homeomorphic.*

Using this theorem, we can prove the following important result.

Theorem 1.7.32. *If X is an infinite dimensional normed space, then $\partial B_1 = \{u \in X : \|u\| = 1\}$ is an* AR *and a retract.*

Proof. By Theorem 1.7.31 X and $X \setminus \{0\}$ are homeomorphic. Due to Corollary 1.7.30, X is an AR. Hence $X \setminus \{0\}$ is an AR as well. Applying the radial retraction $r : X \setminus \{0\} \to \partial B_1$ defined by $r(u) = u/\|u\|$ for all $u \in X \setminus \{0\}$, we see that ∂B_1 is a retract of $X \setminus \{0\}$, hence an AR; see Proposition 1.7.25. Therefore we conclude that ∂B_1 is an AR and a retract of X. ☐

Remark 1.7.33. The result fails if X is finite dimensional. We will show this in Section 6.4 by using fixed point theory.

1.8 Remarks

(1.1) Point set topology emerged as a coherent field of mathematics with Hausdorff's 1914 book [140]. Hausdorff found the right set of axioms to introduce the notion of topology in a general setting. He provided a unified framework for all previous topological research. Abstract spaces were first introduced by Fréchet [117] and Riesz [240]. The notion of a subbasis (see Definition 1.1.3) is due to Bourbaki [42]. The books of Choquet [65], Dugundji [91], Kelley [172], Kuratowski [183, 184], Munkres [226], Nagata [228], and Willard [309] are excellent references for all topics of point-set topology discussed here.

(1.2) The Hausdorff property (see Definition 1.2.1) was among the axioms for a topology used by Hausdorff. Before Hausdorff spaces, there was a more general class, the T_1-spaces introduced by Fréchet and Riesz.

Definition 1.8.1. A topological space X is a **T_1-space** if and only if for every distinct $x, u \in X$, there is a neighborhood of each not containing the other.

Remark 1.8.2. In such spaces singletons are closed sets.

Regular spaces (see Definition 1.2.7) were introduced by Vietoris [294] and the normality property is due to Tietze [285]. Many authors define regularity and normality of T_1-spaces (see Definition 1.8.1): for example, Kelley [172] and Munkres [226]. Here we follow Dugundji [91]. Urysohn's Lemma (see Theorem 1.2.17) was proven by Urysohn [289]. The companion Theorem 1.2.17 (Tietze Extension Theorem) was proven by Tietze [284]. The notion of complete regularity (see Definition 1.2.19) is due to Urysohn [289].

The notions of first and second countability (see Definition 1.2.20) were defined by Hausdorff [140] while the notion of separability is due to Fréchet [117]. The Lindelöf property (see Definition 1.2.26(b)) goes back to Lindelöf [200] for Euclidean spaces. The general study of Lindelöf spaces started with the paper of Kuratowski–Sierpinski [182].

E. H. Moore [219] and E. H. Moore–Smith [220] developed the general theory of convergence using nets, although the term is due to Kelley [171]. Subnets (see Definition 1.2.38) were introduced by E. H. Moore [221] and studied in detail by Kelley [171]. There is an alternative approach using filters instead of nets. This approach is used by Bourbaki [45].

(1.3) Weak topologies are discussed in Bourbaki [45] under the name "initial topologies." Moreover, quotient topologies were first studied by Alexandrov [4] and R. L. Moore [223]. Weak topologies are important in Banach space theory.

(1.4) The notion of connectedness (see Definition 1.4.23(b)) is even older and appears in the work of Weierstraß. Locally connected spaces (see Definition 1.4.34) were introduced by Hahn [135] and are discussed in detail in the books of Dugundji [91] and Kuratowski [184].

Here is another notion of "connectedness" for metric spaces that can traced back to the work of Cantor.

Definition 1.8.3. A metric space (X, d) is said to be **well-chained** (or **well-linked**) if for every pair $(x, u) \in X \times X$ and every $\varepsilon > 0$ there exists a finite sequence $v_1, \ldots, v_n$ of points in X such that $v_1 = x$, $v_n = u$ and $d(v_k, v_{k+1}) \leq \varepsilon$ for all $k \in \{1, \ldots, n-1\}$. That means x and u can be joined by a chain of steps at most equal to ε.

Proposition 1.8.4. *Every connected metric space is well-chained. For compact metric spaces we have "connected $\Longleftrightarrow$ well-chained."*

The term "compact space" is due to Fréchet [117] who used it to describe sequential compactness of metric spaces. Hausdorff [140] observed that the sequential definition of compactness is equivalent to the general definition (see Definition 1.4.42) for metric spaces. Alexandrov–Urysohn [5] used Definition 1.4.42 to describe compact spaces and called them "bicompact spaces." The Product Theorem of Tychonoff (see Theorem 1.4.56) was proven by Tychonoff [288] and showed that Definition 1.4.42 is the right one, that is, more general for compactness since it passes to arbitrary products.

Local compactness was introduced by Alexandrov [3] and Tietze [285]. For a topological vector space, local compactness is equivalent to finite dimensionality.

Local compactness is important in integration theory and in the theory of topological groups.

The problem of compactification was initiated by Alexandrov [3] who introduced the one-point compactification; see Definition 1.4.74. Paracompactness was defined by Dieudonne [81] with important contributions of Michael [213, 215], [216].

(1.5) The extension of topological considerations beyond the realm of Euclidean spaces was achieved by Fréchet [117] who introduced metric spaces and allowed the "points" under consideration to be abstract objects and not real numbers or real vectors. The idea of completion of metric spaces can be traced back to Cauchy who tried to define irrational numbers as the limits of Cauchy sequences of rational numbers. The notion of complete metric space can be found in Fréchet [117] and the general completion construction is due to Hausdorff [140]. The supremum metric (see Definition 1.5.28) although attributed to Fréchet, was first used by Weierstraß back in 1885. The systematic study of continuous maps and homeomorphisms started with Fréchet [117] although the idea of homeomorphism (but in a less general context) was used by Poincaré back in 1895.

Next we present an important theorem that gives us conditions under which a Hausdorff topological space is metrizable. The result is due to Urysohn [290] and is known as the "Urysohn Metrization Theorem."

Theorem 1.8.5 (Urysohn Metrization Theorem). *Every second countable regular topological space is metrizable.*

Polish spaces are discussed in Bourbaki [45] and Souslin spaces in L. Schwartz [268]. More about them in the Remarks of Chapter 2.

The notions of first and second category spaces (see Definition 1.5.62(b),(c)) were introduced by Baire [20] who also proved Theorem 1.5.68(b). Theorem 1.5.68(a) is due to R. L. Moore [222] and Theorem 1.5.69 is due to A. H. Stone [278].

(1.6) The compact-open topology (see Definition 1.6.1) was defined and studied in detail by Arens [10] and Fox [116]. The Arzela–Ascoli Theorem (see Theorem 1.6.14) was first proven for $C[0, 1]$ by Arzela [11] (the necessary part) and by Ascoli [12] (the sufficient part).

Definition 1.8.6. A Hausdorff topological space X is a **k-space** (or a **compactly generated space**) if the following condition hold:

"$C \subseteq X$ is closed if and only if $C \cap K$ is closed for every $K \subseteq X$ compact."

Theorem 1.8.7. (a) *Every locally compact space is a k-space.*
(b) *Every first countable space is a k-space.*

Remark 1.8.8. In particular a metric space is a k-space.

This leads us to the following generalization of Theorem 1.6.14.

Theorem 1.8.9. *Theorem* 1.6.14 *remains true if Y is only a k-space (not necessarily metric space).*

In this general form the result is due to Kelley [172, pp. 233-234].

(1.7) For further results on semicontinuous functions we refer to Dal Maso [70].

The next notion is important in variational problems.

Definition 1.8.10. A function $\varphi\colon X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is said to be **coercive** (**sequentially coercive**) if for every $\lambda \in \mathbb{R}$ the sublevel set $\varphi^\lambda = \{x \in X\colon \varphi(x) \le \lambda\}$ is relatively compact (relatively sequentially compact).

Remark 1.8.11. Sequentially coercivity implies coercivity. Another name for coercivity is **inf-compactness** (**sequential inf-compactness**). Note that lower semicontinuity and coercivity are antagonistic notions. More precisely, let τ_1, τ_2 be two Hausdorff topologies on X and assume that $\tau_2 \subseteq \tau_1$. Then for a function $\varphi\colon X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ we have that "φ is τ_2-lower semicontinuous" implies "φ is τ_1-lower semicontinuous" as well as "φ is τ_1-coercive" implies "φ is τ_2-coercive."

A balance between these two properties leads to the choice of a good topology for variational analysis. For additional information on retracts, absolute retracts, homotopies, etc. we refer to Borsuk [40], Hu [159] and Granas–Dugundji [133].

Problems

Problem 1.1. Suppose that X, Y are Hausdorff topological spaces and $f\colon X \to Y$ is a continuous map. Show that the set $C = \{(x, u) \in X \times X\colon f(x) = f(u)\}$ is closed in $X \times X$ with the product topology.

Problem 1.2. Suppose that X, Y are Hausdorff topological spaces and $f, g\colon X \to Y$ are continuous maps. Show that $\{x \in X\colon f(x) = g(x)\}$ is closed in X.

Problem 1.3. Show that every subspace of a completely regular space is completely regular. Moreover show that $X = \prod_{\alpha\in I} X_\alpha$ with the product topology is completely regular if and only if each factor space X_α is completely regular.

Problem 1.4. Show that X is completely regular if and only if it is homeomorphic to a subspace of some cube.

Problem 1.5. Show that a topological space X is Hausdorff if and only if the diagonal $D = \{(u, u) \in X \times X\colon u \in X\}$ is closed in $X \times X$ with the product topology.

Problem 1.6. Suppose that X is a Hausdorff topological space and let $\{u_n\}_{n\geq 1} \subseteq X$ be a sequence such that $u_n \to u \in X$. Show that the set $K = \{u_n\}_{n\geq 1} \cup \{u\}$ is compact. Is the result true for nets? Justify your answer.

Problem 1.7. Show that a regular Lindelöf space is normal.

Problem 1.8. Suppose that X, Y are Hausdorff topological spaces, Y is compact and $f\colon X \to Y$. Show that f is continuous if and only if $\operatorname{Gr} f = \{(u, y) \in X \times Y\colon y = f(u)\}$ is closed in $X \times Y$ with the product topology.

Problem 1.9. Suppose that $\{X_\alpha\}_{\alpha\in I}$ are Hausdorff topological spaces and $K_\alpha \subseteq X_\alpha$ with $\alpha \in I$ are compact sets. Let $U \subseteq X = \prod_{\alpha\in I} X_\alpha$ be an open set for the product topology such that $\prod_{\alpha\in I} K_\alpha \subseteq U$. Show that there exists a basic open set V (for the product topology) such that $\prod_{\alpha\in I} K_\alpha \subseteq V \subseteq U$.

Problem 1.10. Let X, Y be Hausdorff topological spaces and let $f: X \to Y$ be a map with $\operatorname{Gr} f = \{(u, y) \in X \times Y : y = f(u)\}$, which is closed in $X \times Y$ with the product topology. Show that for every compact $K \subseteq Y$, $f^{-1}(K) \subseteq X$ is closed.

Problem 1.11. Let X be a locally compact topological space. Show that X is second countable if and only if it is separable and metrizable.

Problem 1.12. Let X, Y be Hausdorff topological spaces and $A \subseteq X$, $B \subseteq Y$ are nonempty sets. Show that $A \times B$ is closed (resp. open, dense) in $X \times Y$ with the product topology if and only if A and B are closed (resp. open, dense) in X and Y, respectively.

Problem 1.13. Suppose that X is a normal topological space and $A \subseteq X$ closed. Show that the following statements are equivalent:
(a) A is a G_δ-set.
(b) There exists a continuous map $f: X \to Y$ such that $A = f^{-1}(0)$.
(c) For every closed $C \subseteq X$ with $A \cap C = \emptyset$, there exists a continuous function $f: X \to [0, 1]$ such that $f^{-1}(0) = A$ and $f(B) = 1$.

Problem 1.14. Let (X, τ) be a Hausdorff topological space and $\mathcal{L} \subseteq \tau$ be a subbasis of the topology. Assume that every $\mathcal{L}$-cover of X admits a finite subcover. Show that (X, τ) is compact. Remark: this result is known as “Alexandrov’s Subbasis Theorem.”

Problem 1.15. Let (X, d) be a metric space. Show that there exists a normed space V and an isometry $\xi: X \to V$ such that $\xi(X) \subseteq V$ is closed. Remark: this result is known as the “Arens–Eells Embedding Theorem.”

Problem 1.16. Let $A \subseteq \mathbb{R}^N$ be connected and let $A_\varepsilon = \{u \in \mathbb{R}^N : d(u, A) < \varepsilon\}$. Show that A_ε is connected and path-connected.

Problem 1.17. Let X be a Hausdorff topological space that is connected and A is a proper nonempty subset of X. Show that $\operatorname{bd} A \neq \emptyset$.

Problem 1.18. Let X be a Hausdorff topological space that is connected and $A \subseteq X$. Assume that $\operatorname{bd} A$ is connected. Show that $\overline{A}$ is connected as well.

Problem 1.19. Let X be a Hausdorff topological space and $A \subseteq X$ a connected set. Consider a set $D \subseteq X$ such that $A \cap D \neq \emptyset$ and $A \cap (X \setminus D) \neq \emptyset$. Show that $A \cap \operatorname{bd} D \neq \emptyset$.

Problem 1.20. Let X be a Hausdorff topological space, $\{K_\alpha\}_{\alpha \in I}$ is a family of compact subsets of X and $U \subseteq X$ is an open set such that $\cap_{\alpha \in I} K_\alpha \subseteq U$. Show that there exists a finite $F \subseteq I$ such that $\bigcap_{\alpha \in F} K_\alpha \subseteq U$.

Problem 1.21. Let X be a compact topological space and (Y, d) a metric space. On $C(X, Y)$ we consider the supremum metric d_∞; see Definition 1.5.28. Show that $C(X, Y)$ is d_∞-complete if and only if Y is d-complete.

Problem 1.22. Show that a compact metric space cannot be isometric to a proper subset of itself.

Problem 1.23. Let (X, d) be a compact metric space. Show that:
(a) Every nonexpansive map $f\colon X \to X$ (see Remark 1.5.23) is an isometry.
(b) If $f\colon X \to X$ satisfies $d(x, u) \le d(f(x), f(u))$ for all $x, u \in X$, then f is an isometry.

Problem 1.24. Let X be a noncompact, locally compact Hausdorff topological space and $\hat{X}$ is its one-point Alexandrov compactification; see Theorem 1.4.75. Show that $\hat{X}$ is metrizable if and only if X is second countable.

Problem 1.25. Let (X, d) and (Y, ρ) be two metric spaces. Show the following two statements:
(a) If $f\colon X \to Y$ is continuous, then there exists an equivalent metric $\hat{d}$ on X such that $f\colon (X, \hat{d}) \to (Y, \rho)$ is Lipschitz continuous.
(b) If $\mathcal{L}$ is a countable family of continuous functions from X into Y, then there exists an equivalent metric $\hat{d}$ on X and an equivalent metric $\hat{\rho}$ on Y such that each $f \in \mathcal{L}$ with $f\colon (X, \hat{d}) \to (Y, \hat{\rho})$ is Lipschitz continuous.

Problem 1.26. Let X be a Hausdorff topological space, (Y, d) a metric space, $f\colon X \to Y$ a continuous map, and $D_f = \{x \in X\colon f \text{ is not continuous at } x\}$. Show that D_f is an F_σ-set.

Problem 1.27. Is there a function $f\colon [0, 1] \to \mathbb{R}$ with D_f being the irrational numbers in $[0, 1]$ (see Problem 1.26)? Justify your answer.

Problem 1.28. Let X be a Hausdorff topological space and $\varphi\colon X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ a coercive and lower semicontinuous (resp. sequentially coercive and sequentially lower semicontinuous) function. Show that there exists $u_0 \in X$ such that $\varphi(u_0) = \inf[\varphi(u)\colon u \in X]$.

Problem 1.29. Let $\varphi\colon \mathbb{R}^N \to \mathbb{R}$ be a function such that $\lim_{|u|\to\infty} \varphi(u)/|u| > 0$. Show that φ is coercive in the sense of Definition 1.8.10.

Problem 1.30. Let X, Y be metrizable spaces with Y compact and $\varphi\colon X \times Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ lower semicontinuous. Let $m(u) = \inf[\varphi(u, y)\colon y \in Y]$. Show that $m\colon X \to \overline{\mathbb{R}}$ is lower semicontinuous and for every $u \in X$ there exists $y_0 \in Y$ such that $m(u) = \varphi(u, y_0)$.

Problem 1.31. Suppose that X is a k-space (see Definition 1.8.6) and Y is a Hausdorff topological space. Show that $f\colon X \to Y$ is continuous if and only if $f|_K$ is continuous for every compact $K \subseteq X$.

Problem 1.32. Let X be a metric space, $A \subseteq X$ closed, and $V \subseteq [0, 1] \times X$ an open set such that $[0, 1] \times A \subseteq V$. Show that there exists an open set $U \subseteq X$ such that $A \subseteq U$ and $[0, 1] \times U \subseteq V$.

Problem 1.33. Let X be a Hausdorff topological space and $A \subseteq X$ closed. Show that A is a deformation retract of X if and only if A is a retract of X and X is deformable into A.

Problem 1.34. Let X be an AR. Show that any open set $U \subseteq X$ is also an AR.

Problem 1.35. Show that $\mathbb{Q}$ is not topologically complete.

Problem 1.36. Let

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}\,, \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

being the characteristic function of the rationals. Show that $\chi_{\mathbb{Q}}$ is not the pointwise limit of a sequence of continuous functions.

Problem 1.37. Let (X, d) be a compact metric space and $f: X \to X$ an isometry. Show that f is surjective.

Problem 1.38. Is the pointwise limit of lower semicontinuous functions a lower semicontinuous function? How about the uniform limit? Justify your answer.

Problem 1.39. Show that the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is topologically complete.

Problem 1.40. Let X, Y be Hausdorff topological spaces and $f: X \to Y$. Show that $\operatorname{Gr} f = \{(u, y) \in X \times Y : y = f(u)\}$ is a retract of $X \times Y$.

Problem 1.41. Let (X, τ) be a compact topological space and suppose that there exists a countable, separating family $\mathcal{F}$ of continuous functions $f: X \to Y$ with (Y, d) a metric space. Show that τ is metrizable.

Problem 1.42. Show that every locally compact Souslin space is Polish.

Problem 1.43. Let (X, d) be a metric space and $C_1, C_2 \subseteq X$ nonempty, disjoint, closed sets with C_2 compact. Show that $d(C_1, C_2) = \inf[d(u, v): u \in C_1, v \in C_2] > 0$.

Problem 1.44. Let X be a locally compact and σ-compact topological space. Show that every open cover $\mathcal{L}$ of X has a locally finite open refinement $\{V_n\}_{n\geq 1}$ such that $\overline{V}_n$ is compact for all $n \in \mathbb{N}$.

Problem 1.45. Let X be a metrizable, locally compact, σ-compact topological space. Show the following:

(a) Every open set $U \subseteq X$ can be written as $U = \bigcup_{n\geq 1} K_n$ with compact K_n and $K_n \subseteq \operatorname{int} K_{n+1}$ for all $n \in \mathbb{N}$.

(b) Every compact set $K \subseteq X$ can be written as $K = \bigcap_{n\geq 1} U_n$ with U_n open, $\overline{U}_n$ compact and $U_n \supseteq U_{n+1}$ for all $n \in \mathbb{N}$.

Problem 1.46. Let X be a locally compact space and $\hat{X}$ its one-point Alexandrov compactification. Set $V = \{\hat{f} \in C(\hat{X}, \mathbb{R}): f(\infty) = 0\}$. For every $\hat{f} \in V$, let $\tilde{f} = \hat{f}|_X$. Show that $\hat{f} \to \tilde{f}$ is an isometry of V onto $C_0(X, \mathbb{R}) = \{f \in C(X, \mathbb{R}):$ for every $\varepsilon > 0$ there exists compact $K \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \in X \setminus K\}$ being the space of continuous functions on X vanishing at infinity.

Problem 1.47. Let X, Y be Hausdorff topological spaces, $\{V_\alpha\}_{\alpha\in I}$ an open cover of Y, and $f: X \to Y$ a continuous map such that $f_\alpha = f|_{f^{-1}(V_\alpha)}: f^{-1}(V_\alpha) \to V_\alpha$ is a homeomorphism for every $\alpha \in I$. Show that f is a homeomorphism.

Problem 1.48. Let X, Y be Hausdorff topological space, $f: X \to Y$ a map, and $G = \operatorname{Gr} f = \{(u,y) \in X \times Y : y = f(u)\}$. Let $g: X \to G$ be defined by $g(u) = (u, f(u))$. Show that f is continuous if and only if g is a homeomorphism.

Problem 1.49. Let X be a Baire space, Y a separable metric space and $f: X \to Y$ a map such that the inverse image of any open set is a F_σ-set. Show that f is continuous at every point of a dense G_δ-set.

Problem 1.50. Let X be a second countable regular topological space and $U \subseteq X$ an open set. Show that there exists a continuous function $f: X \to [0,1]$ such that $f(u) > 0$ for all $u \in U$ and $f(u) = 0$ for all $u \in X \setminus U$.

Problem 1.51. Let X, Y be Hausdorff topological spaces, $f: X \to Y$ a continuous map, $C_n \subseteq X$ closed for all $n \in \mathbb{N}$, $C_n \searrow C$ being nonempty compact, and for every $U \supseteq C$ open, there is $n \in \mathbb{N}$ such that $C_n \subseteq U$. Show that $f(C) = \bigcap_{n\geq 1} f(C_n) = \bigcap_{n\geq 1} \overline{f(C_n)}$.

Problem 1.52. Let X be a regular topological space, $K \subseteq X$ compact and $U \subseteq X$ open such that $K \subseteq U$. Show that there exists an open set $V \subseteq X$ such that $K \subseteq V \subseteq \overline{V} \subseteq U$.

Figure 1.2 shows the relations between various spaces introduced in this chapter.

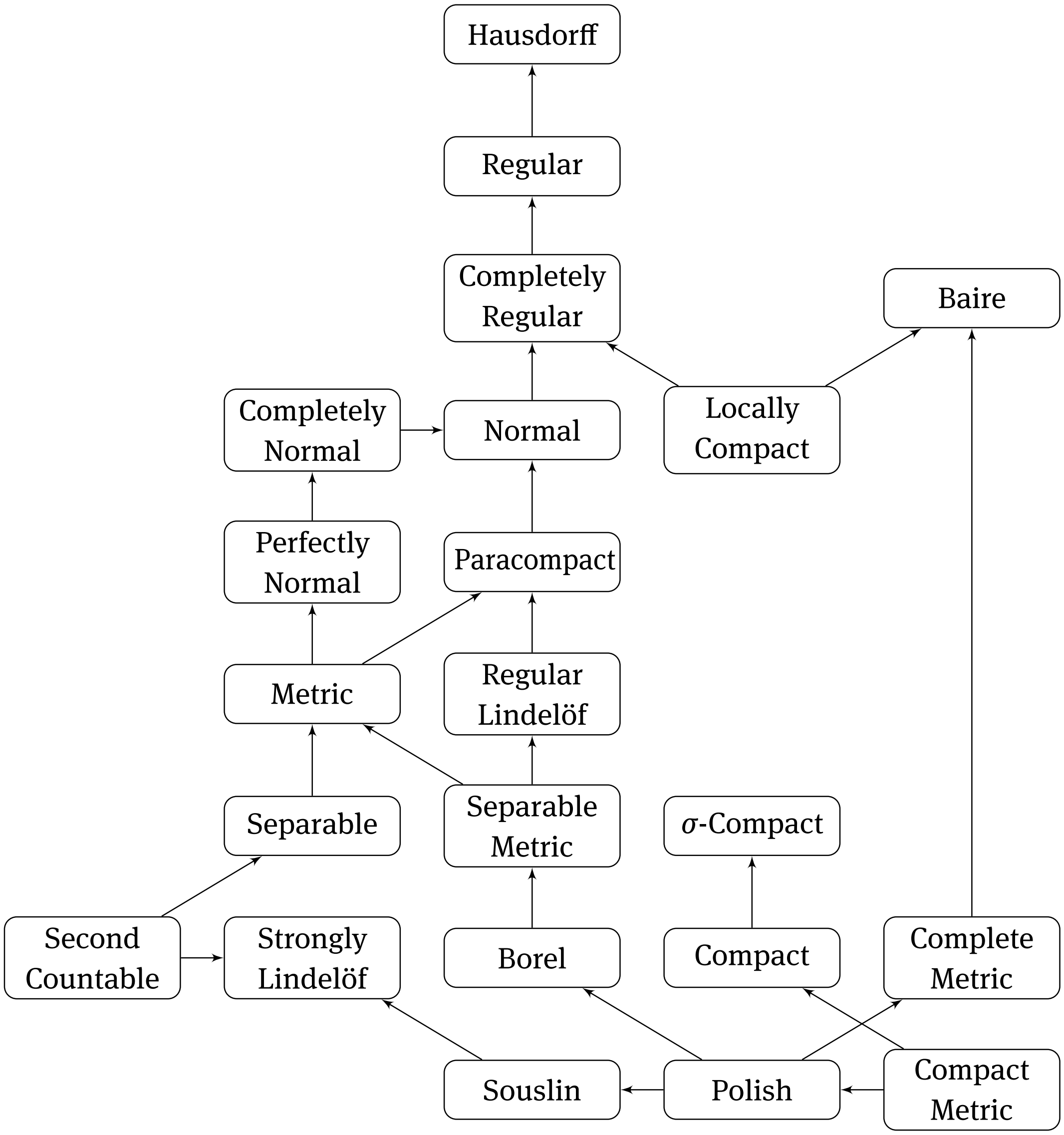

Fig. 1.2: Topological spaces: From Compact Metric to Hausdorff.