3. Topological vector spaces

3.1 Definitions

Banach spaces, and more generally normed spaces, are endowed with two structures: a **linear structure** and a notion of limits, i.e., a **topology**. Many useful spaces are Banach spaces, and indeed, we saw many examples of those. In certain cases, however, one deals with vector spaces with a notion of convergence that is **not normable**. Perhaps the best example is the space of $C_0^\infty(K)$ functions that are compactly supported inside some open domain K (this space is the basis for the theory of distributions). Another such example is the space of continuous functions $C(\Omega)$ defined on some open set $\Omega \subset \mathbb{R}^n$. In such cases, we need a more general construct that embodies both the vector space structure and a topology, without having a norm. In certain cases, the topology is not only not normable, but even **not metrizable** (there are situations in which it is metrizable but not normable).

We start by recalling basic definition in topological spaces:

Definition 3.1 — Topological space. A **topological space** (מרחב טופולוגי) is a set S with a collection τ of subsets (called the **open sets**) that contains both S and $\varnothing$, and is closed under arbitrary union and finite intersections.

A topological space is the most basic concept of a set endowed with a notion of neighborhood.

Definition 3.2 — Open neighborhood. In a topological space (S,τ), a **neighborhood** (סביבה) of a point x is an open set that contains x. We will denote the collection of all the neighborhoods of x by

$$\mathscr{N}_x = \{U \in \tau \mid x \in U\}.$$

Topological spaces are classified according to certain additional properties that they may satisfy. A property satisfied by many topological spaces concerns the **separation** between points:

Definition 3.3 — Hausdorff space. A topological space is a **Hausdorff space** if distinct points have distinct neighborhoods, i.e., $\forall x \neq y$ there are $U_x \in \mathscr{N}_x$ and $U_y \in \mathscr{N}_y$ such that

$$U_x \cap U_y = \varnothing.$$

The Hausdorff property is required, for example, for limits to be unique.

Definition 3.4 — Base. In a topological space (S, τ), a collection $\tau' \subset \tau$ of open sets is a **base for** τ (בסיס) if every open set is a union of members of τ'.

Bases are useful because many properties of topologies can be reduced to statements about a base generating that topology, and because many topologies are most easily defined in terms of a base that generates them.

■ **Example 3.1** The open balls

$$\mathfrak{B}(x,a) = \{y \in S \mid d(y,x) < a\}$$

form a base for a topology in a metric space. A set is open if and only if it is the union of open balls. ■

Definition 3.5 — Local base. Let (S, τ) be a topological space and let $x \in S$. A collection $\gamma_x \subset \mathscr{N}_x$ is called a **local base at** x (בסיס מקומי) if every neighborhood of x contains an element of γ_x.

It is easy to see that the union of all local bases is a base for the topology. Indeed, let $A \in \tau$. Let $x \in A$. Since A is a neighborhood of x, then by the definition of the local base, there exists a $U_x \in \gamma_x$, such that $U_x \subset A$. Clearly,

$$A = \bigcup_{x \in A} U_x,$$

which proves that $\cup_{x \in S} \gamma_x$ is a base for τ.

Definition 3.6 — Closure. Let (S, τ) be a topological space and let $E \subset S$. The **closure** of E, denoted $\overline{E}$, is the set of all points all of whose neighborhoods intersect E:

$$\overline{E} = \{x \in S \mid \forall U \in \mathscr{N}_x,\ U \cap E \neq \varnothing\}.$$

Definition 3.7 — Induced topology. Let (S,τ) be a topological space and let $E \subset S$. Then $\tau \cap E$ is called the **induced topology** on E.

A key concept in topological spaces is that of the convergence of sequences:

Definition 3.8 — Limit. A sequence (x_n) in a Hausdorff space (S,τ) converges to a **limit** $x \in S$ if every neighborhood of x contains all but finitely many points of the sequence.

So we have two notions: that of a vector space and that of a topological space. We now blend the two together.

Definition 3.9 — Topological vector space. Let $\mathscr{V}$ be a vector space endowed with a topology τ. The pair $(\mathscr{V},\tau)$ is called a **topological vector space** if

① For every point $x \in \mathscr{V}$, the singleton $\{x\}$ is a closed set (namely, $\{x\}^c \in \tau$).
② The vector space operations are continuous with respect to τ.

Comment 3.1 The first condition is not required in all texts.

The second condition means that the mappings:

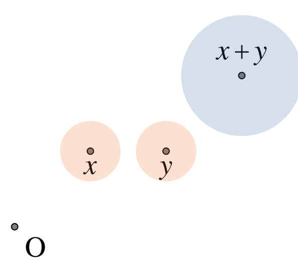
$$\mathscr{V} \times \mathscr{V} \to \mathscr{V} \qquad (x,y) \mapsto x+y$$
$$\mathscr{F} \times \mathscr{V} \to \mathscr{V} \qquad (\alpha,x) \mapsto \alpha x$$

are continuous. That is, for every $x,y \in \mathscr{V}$,

$$(\forall V \in \mathscr{N}_{x+y})(\exists V_x \in \mathscr{N}_x, V_y \in \mathscr{N}_y) : (V_x + V_y \subset V),$$

and for every $x \in \mathscr{V}$ and $\alpha \in \mathscr{F}$,

$$(\forall V \in \mathscr{N}_{\alpha x})(\exists V_x \in \mathscr{N}_x, V_\alpha \in \mathscr{N}_\alpha) : (V_\alpha \cdot V_x \subset V).$$



Having defined a topological vector space, we proceed to define notions concerning subsets:

Definition 3.10 — Bounded set. Let $\mathscr{X}$ be a topological vector space. A subset $E \subset \mathscr{X}$ is said to be **bounded** if

$$(\forall V \in \mathscr{N}_0)(\exists s \in \mathbb{R}) : (\forall t > s)(E \subset tV).$$

That is, every neighborhood of zero contains E after being blown up sufficiently.

As will be shown later, the topological notion of boundedness may not coincide with the metric notion of boundedness.

Definition 3.11 — Balanced set. Let $\mathscr{X}$ be a topological vector space. A subset $E \subset \mathscr{X}$ is said to be **balanced** (מאוזן) if

$$\forall \alpha \in \mathscr{F}, \ |\alpha| \le 1 \qquad \alpha E \subset E.$$

We denote the set of balanced neighborhoods of zero by $\mathscr{N}_0^{\text{bal}}$.

The notion of being balanced is purely algebraic. When we talk about balanced neighborhoods we connect the algebraic concept to the topological concept. Note that in $\mathbb{C}$ the only balanced sets are discs and the whole of $\mathbb{C}$.

Definition 3.12 — Symmetric set. Let $\mathscr{X}$ be a topological vector space. A subset $E \subset \mathscr{X}$ is said to be **symmetric** if

$$x \in E \qquad \text{implies} \qquad (-x) \in E,$$

namely $(-E) = E$. We denote the set of symmetric neighborhoods of x by $\mathscr{N}_x^{\text{sym}}$.

Symmetry is also an algebraic concept. Every balanced set is symmetric; the opposite is not true.

Definition 3.13 With every $a \in \mathscr{X}$ we associate a **translation operator**, $T_a : \mathscr{X} \to \mathscr{X}$, defined by

$$T_a x = x + a,$$

and with every $0 \ne \alpha \in \mathscr{F}$ we associate a **multiplication operator**, $M_\alpha : \mathscr{X} \to \mathscr{X}$, defined by

$$M_\alpha x = \alpha x.$$

Proposition 3.1 Both T_a and M_α are homeomorphisms of $\mathscr{X}$ onto $\mathscr{X}$.

Proof. Both T_a and M_α are bijections by the vector space axioms. Their inverses are T_{-a} and $M_{1/\alpha}$. All are continuous by the very definition of a topological vector space. ■

Corollary 3.2 Every open set is translationally invariant: $E \subset \mathscr{X}$ is open if and only if $a + E$ is open for every $a \in \mathscr{X}$. In particular,

$$\mathscr{N}_a = a + \mathscr{N}_0.$$

Hence the topology is fully determined by the neighborhoods of the origin.

We conclude this section with a classification of various types of topological vector spaces:

Definition 3.14 Let $(\mathscr{X}, \tau)$ be a topological vector space.

① $\mathscr{X}$ is **locally convex** (**קמור מקומית**) if there exists a local base at 0 whose members are convex.
② $\mathscr{X}$ is **locally bounded** if 0 has a bounded neighborhood.
③ $\mathscr{X}$ is **locally compact** if 0 has a neighborhood whose closure is compact.
④ $\mathscr{X}$ is **metrizable** (**מטריזבילי**) if it is compatible with some metric d (i.e., τ is generated by the open balls $\mathfrak{B}(x,a) = \{y \in \mathscr{X} \mid d(y,x) < a\}$).
⑤ $\mathscr{X}$ is an **F-space** if its topology is induced by a complete translationally invariant metric. Every Banach space is an F-space. An F-space is a Banach space if in addition $d(\alpha x, 0) = |\alpha| d(x, 0)$.
⑥ $\mathscr{X}$ is a **Frechet space** if it is a locally convex F-space.
⑦ $\mathscr{X}$ is **normable** (**נורמבילי**) if it can be endowed with a norm whose induced metric is compatible with τ.
⑧ $\mathscr{X}$ has the **Heine-Borel property** if every closed and bounded set is compact.

As we will see, local convexity is important because local convexity amount to the topology being generated by a family of seminorms. Local convexity is also the minimum requirement for the validity of geometric Hahn-Banach properties. Weak topologies, which we will investigate later are always locally convex. We will prove that the only topological vector spaces that are locally compact are finite dimensional. We will prove that a topological vector space is metrizable if it has a countable local base at the origin, which in turn, is guaranteed if the space is locally bounded. We will prove that a topological vector space is normable if and only if it is both locally convex and locally bounded.

Exercise 3.1 Consider the vector space $\mathbb{R}$ endowed with the topology τ generated by the base

$$\mathscr{B} = \{[a,b) \mid a < b\}.$$

Show that $(\mathbb{R},\tau)$ is not a topological vector space. ■

3.2 Separation theorems

A topological vector space can be quite abstract. All we know is that there is a vector space structure and a topology that is compatible with it. We have to start make our way from these very elementary concepts.

Lemma 3.3 Let $\mathscr{X}$ be a topological vector space.

$$\forall W \in \mathscr{N}_0 \quad \exists U \in \mathscr{N}_0^{\text{sym}} \quad \text{such that} \quad U + U \subset W.$$

Proof. Since $0+0=0$ and addition is continuous, there exist neighborhoods $V_1, V_2 \in \mathscr{N}_0$ such that

$$V_1 + V_2 \subset W.$$

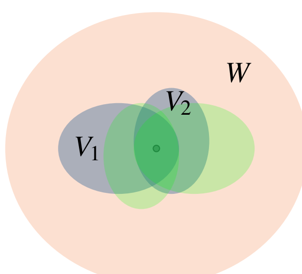
Set

$$U = V_1 \cap (-V_1) \cap V_2 \cap (-V_2).$$

U is symmetric; it is an intersection of four open sets that contain zero, hence it is a non-empty neighborhood of zero. Since $U \subset V_1$ and $U \subset V_2$ it follows that

$$U + U \subset W.$$

■



Lemma 3.4 Let $\mathscr{X}$ be a topological vector space.

$$\forall W \in \mathscr{N}_0 \quad \exists V \in \mathscr{N}_0^{\text{sym}} \quad \text{such that} \quad V + V + V + V \subset W.$$

Proof. Apply the previous lemma with U in place of W. ■

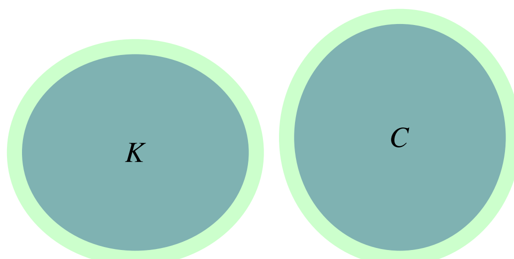
Theorem 3.5 Let $\mathscr{X}$ be a topological vector space. Let $K, C \subset \mathscr{X}$ satisfy:

$$K \text{ is compact}, \quad C \text{ is closed} \quad \text{and} \quad K \cap C = \varnothing.$$

Then there exists a $V \in \mathscr{N}_0$ such that

$$(K + V) \cap (C + V) = \varnothing.$$

In other words, there exist disjoint open sets that contain K and C.



Comment 3.2 Thus, a topological vector space is **regular** (a topological space is regular if separates points from closed sets that do not include that point).

Proof. Let $x \in K$. Since C^c is an open neighborhood of x, it follows from the above lemma that there exists a $V_x \in \mathscr{N}_0^{\text{sym}}$ such that

$$x + V_x + V_x + V_x \in C^c,$$

i.e.,

$$(x + V_x + V_x + V_x) \cap C = \varnothing.$$

Since V_x is symmetric,

$$(x + V_x + V_x) \cap (C + V_x) = \varnothing.$$

For every $x \in K$ corresponds such a V_x. Since K is compact, there exists a finite collection of $(x_i, V_{x_i})_{i=1}^n$ such that

$$K \subset \bigcup_{i=1}^n (x_i + V_{x_i}).$$

Define

$$V = V_{x_1} \cap \cdots \cap V_{x_n}.$$

Then, for every i,

$$(x + V_{x_i} + V_{x_i}) \quad \text{does not intersect} \quad (C + V_{x_i}),$$

and a forteriori,

$$(x + V_{x_i} + V) \quad \text{does not intersect} \quad (C + V).$$

Taking the union over i:

$$K + V \subset \bigcup_{i=1}^{n} (x_i + V_{x_i} + V) \quad \text{does not intersect} \quad (C + V).$$

■

Comment 3.3 Since $K+V$ and $C+V$ are both open and mutually disjoint, it follows that also

$$\overline{K+V} \quad \text{does not intersect} \quad C+V.$$

Indeed, if $x \in C+V$ then there exists $U \in \mathscr{N}_x$ such that

$$U \subset C + V.$$

But since $x \in \overline{K+V}$, every neighborhood of x, and U in particular, intersects $K+V$, i.e., where exists a $y \in U$ satisfying

$$y \in (K+V) \cap (C+V),$$

which is a contradiction.

Corollary 3.6 Let $\mathscr{B}$ be a local base at zero for a topological vector space $\mathscr{X}$. Then every member in $\mathscr{B}$ contains the closure of some other member in $\mathscr{B}$. That is,

$$\forall U \in \mathscr{B} \quad \exists W \in \mathscr{B} \quad \text{such that} \quad \overline{W} \subset U.$$

Proof. Let $U \in \mathscr{B}$. Let $K = \{0\}$ (compact) and $C = U^c$ (closed). By Theorem 3.5 there exists a $V \in \mathscr{N}_0^{\text{sym}}$, such that

$$V \cap (U^c + V) = \varnothing.$$

It follows that

$$V \subset (U^c + V)^c \subset U.$$

By the definition of a local base there exists a neighborhood $W \in \mathscr{B}$ such that

$$W \subset V \subset (U^c + V)^c \subset U.$$

Since $(U^c + V)^c$ is closed,

$$\overline{W} \subset (U^c + V)^c \subset U.$$

■

Corollary 3.7 Every topological vector space $\mathscr{X}$ is Hausdorff.

Proof. Let $x, y \in \mathscr{X}$, $x \neq y$. Both sets $\{x\}$ and $\{y\}$ are closed (by definition) and compact, hence $\mathscr{X}$ is Hausdorff by Theorem 3.5. ■

The following proposition establishes relations between sets, their closure, and their interior.

Proposition 3.8 Let $\mathscr{X}$ be a topological vector space.

① For $A \subset \mathscr{X}$,

$$\overline{A} = \bigcap_{V \in \mathscr{N}_0} (A + V).$$

That is, the closure of a set is the intersection of all the open neighborhoods of that set.

② $\overline{A} + \overline{B} \subset \overline{A + B}$.

③ If $\mathscr{Y} \subset \mathscr{X}$ is a linear subspace, then so is $\overline{\mathscr{Y}}$.

④ If C is convex so is $\overline{C}$.

⑤ If C is convex so is C°.

⑥ For every $B \subset \mathscr{X}$: if B is balanced so is $\overline{B}$.

⑦ For every $B \subset \mathscr{X}$: if B is balanced and $0 \in B^\circ$ then B° is balanced.

⑧ If $E \subset \mathscr{X}$ is bounded then so is $\overline{E}$.

Proof.

① Let $x \in \overline{A}$. By definition, for every $V \in \mathscr{N}_0$, $x + V$ intersects A, of $x \in A - V$. Thus,

$$x \in \bigcap_{V \in \mathscr{N}_0} (A - V) = \bigcap_{V \in \mathscr{N}_0} (A + V).$$

Conversely, suppose that $x \notin \overline{A}$. Then, there exists a $V \in \mathscr{N}_0$ such that $x + V$ does not intersect A, i.e., $x \notin A - V$, hence

$$x \notin \bigcap_{V \in \mathscr{N}_0} (A + V).$$

② Let $a \in \overline{A}$ and $b \in \overline{B}$. By the continuity of vector addition, for every $U \in \mathscr{N}_{a+b}$ there exist $V \in \mathscr{N}_a$ and $W \in \mathscr{N}_b$ such that

$$V + W \subset U.$$

By the definition of $\overline{A}$ every neighborhood of a intersects A and by the definition of $\overline{B}$ every neighborhood of b intersects W: that is, there exist $x \in V \cap A$ and $y \in W \cap B$. Then,

$$x \in A \quad \text{and} \quad y \in B \quad \text{implies} \quad x + y \in A + B,$$

and

$$x \in V \quad \text{and} \quad y \in W \quad \text{implies} \quad x + y \in V + W \subset U.$$

In other words, every neighborhood of $a + b \in \overline{A} + \overline{B}$ intersects $A + B$, which implies that $a + b \in \overline{A + B}$, and therefore

$$\overline{A} + \overline{B} \subset \overline{A + B}.$$

③ Let $\mathscr{Y}$ be a linear subspace of $\mathscr{X}$, which means that,

$$\mathscr{Y} + \mathscr{Y} \subset \mathscr{Y} \qquad \text{and} \qquad \forall a \in \mathscr{F}, \quad a\mathscr{Y} \subset \mathscr{Y}.$$

By the previous item,

$$\overline{\mathscr{Y}} + \overline{\mathscr{Y}} \subset \overline{\mathscr{Y} + \mathscr{Y}} \subset \overline{\mathscr{Y}}.$$

Since scalar multiplication is a homeomorphism it maps the closure of a set into the closure of its image, namely, for every $a \in \mathscr{F}$,

$$a\overline{\mathscr{Y}} \subset \overline{\mathscr{Y}}.$$

④ Convexity is a purely algebraic property, but closures and interiors are topological concepts. The convexity of C implies that for all $t \in [0,1]$:

$$tC + (1-t)C \subset C.$$

Let $t \in [0,1]$, then

$$t\overline{C} = \overline{tC} \qquad \text{and} \qquad (1-t)\overline{C} = \overline{(1-t)C}.$$

By the second item:

$$t\overline{C} + (1-t)\overline{C} = \overline{tC} + \overline{(1-t)C} \subset \overline{tC + (1-t)C} \subset \overline{C},$$

which proves that $\overline{C}$ is convex.

⑤ Suppose once again that C is convex. Let $x,y \in C^\circ$. This means that there exist neighborhoods $U,V \in \mathscr{N}_0$, such that

$$x+U \subset C \qquad \text{and} \qquad y+V \subset C.$$

Since C is convex:

$$t(x+U)+(1-t)(y+V) = (tx+(1-t)y)+U+(1-t)V \subset C,$$

which proves that $tx+(1-t)y \in C^\circ$, namely C° is convex.

⑥ Since multiplication by a (non-zero) scalar is a homeomorphism,

$$\alpha\overline{B} = \overline{\alpha B}.$$

If B is balanced, then for $|\alpha| \le 1$,

$$\alpha\overline{B} = \overline{\alpha B} \subset \overline{B},$$

hence $\overline{B}$ is balanced.

⑦ Again, for every $0 < |\alpha| \le 1$,

$$\alpha B^\circ = (\alpha B)^\circ \subset B^\circ.$$

Since for $\alpha = 0$, $\alpha B^\circ = \{0\}$, we must require that $0 \in B^\circ$ for the latter to be balanced.

⑧ Let V be a neighborhood of zero. By Corollary 3.6 there exists a neighborhood W of zero such that $\overline{W} \subset V$. Since E is bounded, $E \subset tW \subset t\overline{W} \subset tV$ for sufficiently large t. It follows that for large enough t,

$$\overline{E} \subset t\overline{W} \subset tV,$$

which proves that $\overline{E}$ is bounded.

■

Exercise 3.2 Let

$$A = \{(z_1,z_2) \in \mathbb{C}^2 \mid |z_1| \le |z_2|\} \subset \mathbb{C}^2.$$

Show that A is balanced by A° is not. ▪

Theorem 3.9 Let $\mathscr{X}$ be a topological vector space.

① Every neighborhood of zero contains a balanced neighborhood of zero:

$$\forall U \in \mathscr{N}_0 \quad \exists W \in \mathscr{N}_0^{\text{bal}} \quad \text{such that} \quad W \subset U.$$

② Every convex neighborhood of zero contains a balanced convex neighborhood of zero:

$$\forall U \in \mathscr{N}_0^{\text{conv}} \quad \exists A \in \mathscr{N}_0^{\text{bal,conv}} \quad \text{such that} \quad A \subset U.$$

Proof.

① Let $U \in \mathscr{N}_0$. Since scalar multiplication is continuous, and $0 \cdot 0 = 0$, there exists a $V \in \mathscr{N}_0$ and a disc

$$D = \{\alpha \in \mathbb{C} \mid |\alpha| < \delta\},$$

such that

$$DV \subset U.$$

It is easy to see that DV is balanced (thus, the existence of a balanced sub-neighborhood in $\mathscr{X}$ follows from the existence of a balanced sub-neighborhood in the field of scalars).

② Let $U \in \mathscr{N}_0^{\text{conv}}$ and set

$$A = \bigcap_{|\alpha|=1} \alpha U.$$

Since A is the intersection of convex sets it is convex. It is balanced because for every $|\beta| \le 1$,

$$\beta A = \bigcap_{|\alpha|=1} (\beta\alpha) U = \bigcap_{|\alpha|=1} (|\beta|\alpha) U = |\beta| A,$$

and by convexity,

$$|\beta| A = |\beta| A + (1 - |\beta|)\{0\} \subset A.$$

Since A contains the origin, A° is balanced; it is also convex.

■

Corollary 3.10 Every topological vector space has a local base whose elements are balanced (a balanced local base).

Corollary 3.11 Every locally convex topological vector space has a local base whose elements are convex and balanced (a balanced convex local base).

Theorem 3.12 Let $V \in \mathscr{N}_0$ in a topological vector space $\mathscr{X}$. Then:

① For every sequence $r_n \to \infty$,

$$\bigcup_{n=1}^{\infty} r_n V = \mathscr{X}.$$

② Every compact set is bounded.

③ If V is bounded (which requires $\mathscr{X}$ to be locally bounded) and $\delta_n \to 0$, then

$$\{\delta_n V \mid n \in \mathbb{N}\}$$

is a local base for $\mathscr{X}$.

Proof.

① Let $x \in \mathscr{X}$ and consider the sequence x/r_n. This sequence converge to zero by the continuity of the scalar multiplication $\mathscr{F} \times \mathscr{X} \to \mathscr{X}$. Thus, for sufficiently large n,

$$x/r_n \in V \qquad \text{i.e.} \qquad x \in r_n V.$$

② Let $K \subset \mathscr{X}$ be compact. We need to prove that it is bounded, namely, that for every $V \in \mathscr{N}_0$,

$$K \subset tV \quad \text{for sufficiently large } t.$$

Let $V \in \mathscr{N}_0$ be given. By Theorem 3.9 there exists a $W \in \mathscr{N}_0^{\text{bal}}$ such that $W \subset V$. By the previous item

$$K \subset \bigcup_{n=1}^{\infty} (nW).$$

Since K is compact,

$$K \subset \bigcup_{k=1}^{K} (n_k W) = n_K \bigcup_{k=1}^{K} \left(\frac{n_k}{n_K} W \right) \subset n_K W,$$

where in the last step we used the fact that W is balanced. Thus,

$$\forall t > n_K \qquad K \subset n_K W = t \left(\frac{n_K}{t} W \right) \subset tW, \subset tW \subset tV,$$

which proves that K is bounded.

③ Let $U \in \mathscr{N}_0$. Since V is bounded,

$$V \subset tU \quad \text{for sufficiently large } t,$$

or,

$$t^{-1} V \subset U. \quad \text{for sufficiently large } t.$$

Thus, $\delta_n V \subset U$ for sufficiently large n.

■

Exercise 3.3 Let $(\mathscr{X}, \tau)$ be a topological vector space. Show that a set $E \subset \mathscr{X}$ is bounded if and only if every countable subset of E is bounded. ■

Exercise 3.4 Let $(\mathscr{X}, \tau)$ be a topological vector space. Show that:

① If $\mathscr{X}$ is locally bounded then the convex hull of a convex set is bounded.
② If A, B are bounded so is $A + B$.
③ If A, B are compact so is $A + B$.
④ if A is compact and B is closed then $A + B$ is closed.
⑤ Give an example where A, B are closed by $A + B$ is not closed.

■

3.3 Linear maps

We already know what are linear maps between vector spaces. The following theorem is an obvious consequence of the translational invariance of the space:

Theorem 3.13 Let $\mathscr{X}$ and $\mathscr{Y}$ be topological vector spaces and let $T : \mathscr{X} \to \mathscr{Y}$ be linear. If T is continuous at zero then it is continuous everywhere.

The following Lemma will be needed below:

Lemma 3.14 Linear maps between topological vector spaces map balanced sets into balanced sets.

Proof. Let $T : \mathscr{X} \to \mathscr{Y}$ be linear, and let $V \subset X$ be balanced. For every $|\alpha| \le 1$,

$$\alpha\{T(x) : x \in V\} = \{\alpha T(x) : x \in V\} = \{T(\alpha x) : x \in V\} = \{T(y) : y \in \alpha V\} \subset \{T(y) : y \in V\}.$$

■

In Banach spaces continuous and bounded linear maps are the same. We need a notion of bounded map to obtain a generalization for topological vector spaces.

Definition 3.15 A linear map $T : \mathscr{X} \to \mathscr{Y}$ is **bounded** if it maps bounded sets into bounded sets.

The following theorem concerns only scalar-valued linear maps, i.e., linear functionals. Please note that since $\mathscr{F}$ is a topological vector space, there is a well-defined notion of continuity for functionals. In particular, we may define:

Definition 3.16 Let $(\mathscr{X},\tau)$ be a topological vector space. Its **dual** X^* is the space of continuous linear functionals over $\mathscr{X}$.

As of now, $\mathscr{X}^*$ is not endowed with any topology, as it was in Banach spaces. We will see how to topologize $\mathscr{X}^*$ later when we study weak topologies.

The following theorem establishes the equivalence between continuity and boundedness for linear functionals:

Theorem 3.15 Let $0 \neq f : \mathscr{X} \to \mathscr{F}$ be a linear functional. Then the following assertions are equivalent:

① f is continuous.
② $\ker f$ is closed.
③ $\ker f$ is not dense in $\mathscr{X}$.
④ f maps some $V \in \mathscr{N}_0$ into a bounded set in $\mathscr{F}$.

Proof.

① Suppose that f is continuous, then since $\{0\} \in \mathscr{F}$ is closed then $f^{-1}(0) = \ker f$ is closed.

② Suppose that $\ker f$ is closed. Since we assume that $f \neq 0$, it must be that $\ker f \neq \mathscr{X}$, hence $\ker f$ can't be dense in $\mathscr{X}$.

③ Suppose that $\ker f$ is not dense in $\mathscr{X}$. That is, its complement has a non-empty interior. There exists an $x \in \mathscr{X}$ and a $V \in \mathscr{N}_0$, such that

$$(x+V)\cap \ker f = \varnothing. \tag{3.1}$$

This means that

$$f(x+V) \not\ni 0,$$

i.e.,

$$\forall y \in V \qquad f(y) \neq -f(x).$$

By Theorem 3.9 (every neighborhood contains a balanced neighborhood) we may assume that V is balanced; by Lemma 3.14, $f(V)$ is balanced as well.

Balanced set in $\mathbb{C}$ are either bounded, in which case we are done, of equal to the whole of $\mathbb{C}$, which contradicts the requirement that $f(y) \neq -f(x)$ for all $y \in V$.

④ Suppose that $f(V)$ is bounded for some $V \in \mathscr{N}_0$, i.e., there exists an M such that

$$\forall x \in V \qquad |f(x)| \le M.$$

Let $\varepsilon > 0$ be given. Set $W = (\varepsilon/M)V$. Then for all $y \in W$

$$|f(y)| \le \frac{\varepsilon}{M} \sup_{x \in V} |f(x)| \le \varepsilon,$$

which proves that f is continuous at zero (and hence everywhere).

■

Comment 3.4 If f maps some $V \in \mathscr{N}_0$ into a bounded set then f is bounded. Let $W \subset \mathscr{X}$ be bounded. Then for large enough t:

$$W \subset tV,$$

hence

$$\sup|f(W)| \le \sup|f(tV)| = t \sup|f(V)| < \infty.$$

3.4 Finite dimensional spaces

As for Banach spaces, finite-dimensional topological vector spaces are simpler than infinite-dimensional ones.

Lemma 3.16 Let $\mathscr{X}$ be a topological vector space. Any linear function $T : \mathscr{F}^n \to \mathscr{X}$ is continuous.

Proof. Denote by $\{e_j\}$ the standard basis in $\mathscr{F}^n$ and set

$$u_j = T(e_j) \qquad j = 1, \dots, n.$$

By linearity, for any $z = (z_1, \dots, z_n)$,

$$T(z) = \sum_{j=1}^{n} z_j u_j.$$

The map $z \mapsto z_j$ (which is a map $\mathscr{F}^n \to \mathscr{F}$) is continuous and so are addition and scalar multiplication in $\mathscr{X}$. ■

The following proposition shows, in particular, that all finite-dimensional subspaces of a topological vector space are closed.

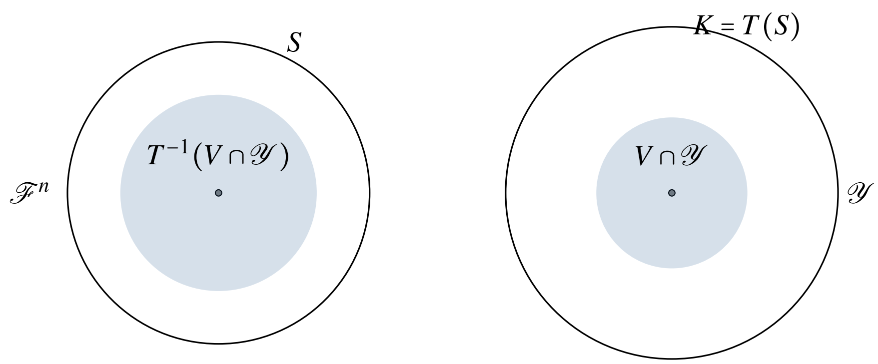
Proposition 3.17 Let $\mathscr{Y}$ be an n-dimensional subspace of a topological vector space $\mathscr{X}$. Then:

① Every isomorphism $\mathscr{F}^n \to \mathscr{Y}$ (equivalence in the category of vector spaces) is a homeomorphism (equivalence in the category of topological spaces).

② $\mathscr{Y}$ is closed in $\mathscr{X}$.

Proof.

① Let $T : \mathscr{F}^n \leftrightarrows \mathscr{Y}$ be an isomorphism (i.e., linear and bijective). We need to show that both T and T^{-1} are continuous. By the previous lemma T is continuous, so only remains T^{-1}.



let $S = \partial\mathfrak{B}_{\mathscr{F}^n}$ be the unit sphere in $\mathscr{F}^n$, and let $K = T(S)$. By Lemma 3.16 since T is linear it is also continuous, and since continuous functions map compact sets into compact sets, K is compact.

Since $0 \notin S$ and f is injective, it follows that $0 \notin K$, and therefore

$$\exists V \in \mathscr{N}_0^{\text{bal}} \quad \text{such that} \quad V \cap K = \varnothing.$$

Note that V is a neighborhood of 0 in $\mathscr{X}$, but $V \cap \mathscr{Y}$ is a neighborhood of zero in $\mathscr{Y}$ (in the induced topology). By injectivity,

$$E = T^{-1}(V \cap \mathscr{Y})$$

is disjoint from S.

By Lemma 3.14, E which is the linear image of a balanced set is also balanced, hence it is connected. It follows that T^{-1} maps $V \cap \mathscr{Y}$ into $\mathfrak{B}_{\mathscr{F}^n}$. T^{-1} is an n-tuple of linear functionals on $\mathscr{Y}$. Since it maps a bounded neighborhood of zero in $\mathscr{Y}$ into a bounded set in $\mathscr{F}$, it is continuous by Theorem 3.15, hence T is a homeomorphism.

② Let $\{u_j\}$ be a basis for $\mathscr{Y}$, and let $T : \mathscr{F}^n \to \mathscr{Y}$ be a linear map defined by $T(e_j) = u_j$. It is an isomorphism, hence by the preceding item it is a homeomorphism. Let as above $K = T(S)$ and let V be a balanced neighborhood of 0 (in $\mathscr{X}$) that does not intersect K.

Let $x \in \overline{\mathscr{Y}}$. By Theorem 3.12, there exists a $t > 0$ for which

$$x \in tV,$$

and therefore it is in the closure of the set[1]

$$tV \cap \mathscr{Y} \subset tT(\mathfrak{B}_{\mathscr{F}^n}) \subset T(t\overline{\mathfrak{B}_{\mathscr{F}^n}}).$$

The latter being compact, it is closed. Since x is in the closure of a closed set,

$$x \in T(t\overline{\mathfrak{B}_{\mathscr{F}^n}}) \subset \mathscr{Y}.$$

■

Corollary 3.18 There is exists a unique topology on $\mathscr{F}^n$ (viewed as a topological vector space), and all n-dimensional topological vector spaces are homeomorphic.

Proposition 3.19 Every locally compact topological vector space has finite dimension.

Proof. Let $\mathscr{X}$ be a locally compact topological vector space: it has some neighborhood V whose closure $\overline{V}$ is compact. By Theorem 3.12 $\overline{V}$ is bounded and hence also V. Moreover,

$$\mathscr{B} = \{V/2^n : n \in \mathbb{N}\}$$

is a local base for $\mathscr{X}$.

Let $y \in \overline{V}$. By definition $y - \frac{1}{2}V$ intersects V, i.e., $y \in V + \frac{1}{2}V$, from which we infer that

$$\overline{V} \subset V + \tfrac{1}{2}V = \bigcup_{x \in V} (x + \tfrac{1}{2}V).$$

Since $\overline{V}$ is compact, it can be covered by a finite union:

$$\overline{V} \subset (x_1 + \tfrac{1}{2}V) \cap \cdots \cap (x_m + \tfrac{1}{2}V).$$

Let $\mathscr{Y} = \operatorname{Span}\{x_1, \dots, x_m\}$. It is a finite-dimensional subspace of $\mathscr{X}$, hence closed by Proposition 3.17. Since $V \subset \mathscr{Y} + \frac{1}{2}V$, and since $\alpha\mathscr{Y} = \mathscr{Y}$ for every $\alpha \neq 0$, it follows that

$$\tfrac{1}{2}V \subset \tfrac{1}{2}\mathscr{Y} + \tfrac{1}{4}V = \mathscr{Y} + \tfrac{1}{4}V,$$

[1] It is easy to show that $x \in V$ and $x \in \overline{\mathscr{Y}}$ implies that $x \in \overline{V \cap \mathscr{Y}}$.

and hence

$$V \subset \mathscr{Y} + \tfrac{1}{2}V \subset \mathscr{Y} + \mathscr{Y} + \tfrac{1}{4}V = \mathscr{Y} + \tfrac{1}{4}V.$$

We may continue similarly to show that $V \subset \mathscr{Y} + \frac{1}{8}V$, and so on, namely

$$V \subset \bigcap_{n=1}^{\infty} (\mathscr{Y} + \tfrac{1}{2^n}V).$$

Since $\mathscr{B}$ is a local base at zero, it follows that V is a subset of every neighborhood of $\mathscr{Y}$, that is[2]

$$V \subset \overline{\mathscr{Y}} = \mathscr{Y}.$$

On the other hand, still by Theorem 3.12,

$$\mathscr{X} = \bigcup_{n=1}^{\infty} nV \subset \mathscr{Y},$$

which implies $\mathscr{X} = \mathscr{Y}$, i.e., $\mathscr{X}$ is finite-dimensional. ∎

Corollary 3.20 Every locally bounded topological vector space that has the Heine-Borel property has finite dimension.

Proof. If $\mathscr{X}$ is locally bounded then it has a bounded neighborhood V. It follows that $\overline{V}$ is also bounded. Since $\mathscr{X}$ satisfies the Heine-Borel property $\overline{V}$ is compact, i.e., $\mathscr{X}$ is locally compact, and therefore has finite dimension. ∎

3.5 Metrization

What does it take for a topological vector space to be metrizable? Suppose there is a metric d compatible with the topology τ. Thus, all open sets are unions of open balls, and in particular, the countable collection of balls $\mathfrak{B}(0,1/n)$ forms a local base at the origin. It turns out that the existence of a countable local base is also sufficient for metrizability.

Theorem 3.21 Let $\mathscr{X}$ be topological vector space that has a countable local base. Then there is a metric d on $\mathscr{X}$ such that:

① d is compatible with τ (every τ-open set is a union of d-open balls).
② The open balls $\mathfrak{B}(0,a)$ are balanced.
③ d is invariant: $d(x+z,y+z) = d(x,y)$.

[2] Indeed, let $x \in V$, then for every n,

$$x \in \mathscr{Y} + \tfrac{1}{2^n}V.$$

Let $U \in \mathscr{N}_0$. Then $x+U$ intersects $\mathscr{Y}$ which proves that $x \in \overline{\mathscr{Y}}$.

④ If, in addition, $\mathscr{X}$ is locally convex, then d can be chosen such that all open balls are convex. (Please note: we are used that open balls are convex; this is true in a normed space but not in general metric spaces.)

Proof. Let $\mathscr{B}'$ be a countable local base. By Theorem 3.9 and Lemma 3.3 there exists a countable local base $\mathscr{B} = \{V_n\}$ whose members are all balanced, and furthermore,

$$V_{n+1} + V_{n+1} + V_{n+1} + V_{n+1} \subset V_n.$$

(If $\mathscr{X}$ is locally convex, the local base can be chosen to include convex sets.) This implies that for all n and k:

$$V_{n+1} + V_{n+2} + V_{n+3} + \cdots + V_{n+k} \subset V_n.$$

Let D be the set of dyadic rational numbers:

$$D = \left\{ \sum_{n=1}^{\infty} \frac{c_n}{2^n} \mid c_n \in \{0,1\}, \quad \text{and } c_n = 0 \text{ for } n > N, N \in \mathbb{N} \right\}.$$

D is dense in $[0,1]$. Define the function $\varphi : D \cup \{r \geq 1\} \to 2^{\mathscr{X}}$:

$$\varphi(r) = \begin{cases} \mathscr{X} & r \geq 1 \\ c_1(r)V_1 + c_2(r)V_2 + \dots & r \in D. \end{cases}$$

The sum in this definition is always finite. For example, $\varphi(1.2) = \mathscr{X}$ and $\varphi(0.75) = V_1 + V_2$. By the property of the base $\{V_n\}$,

$$\varphi\left(\sum_{n=N_1}^{N_2} \frac{c_n}{2^n} \right) = \sum_{n=N_1}^{N_2} c_n V_n \subset V_{N_1 - 1}.$$

Then define the functional $f : \mathscr{X} \to \mathbb{R}$:

$$f(x) = \inf\{r : x \in \varphi(r)\},$$

and define

$$d(x,y) = f(y-x).$$

We will show that d is indeed a metric on $\mathscr{X}$ that satisfies all required properties. This will rely on the following fact:

Lemma 3.22 For $r, s \in D$

$$\varphi(r) + \varphi(s) \subset \varphi(r+s).$$

Proof. If $r+s \geq 1$ then this is obvious as $\varphi(r+s) = \mathscr{X}$.

Suppose then that $r,s \in D$ and $r+s \in D$. The first possibility is that $c_n(r)+c_n(s) = c_n(r+s)$ for all n. This happens if $c_n(r)$ and $c_n(s)$ are never both equal to one. Then,

$$\varphi(r+s) = \sum_{n=1}^{\infty} c_n(r+s)V_n = \sum_{n=1}^{\infty} c_n(r)V_n + \sum_{n=1}^{\infty} c_n(s)V_n = \varphi(r)+\varphi(s).$$

Otherwise, there exists an n for which

$$c_n(r)+c_n(s) \neq c_n(r+s).$$

Let N be the smallest n where this happens: then

$$c_N(r) = c_N(s) = 0 \qquad \text{and} \qquad c_N(r+s) = 1.$$

It follows that

$$\begin{aligned}\varphi(r) &\subset c_1(r)V_1 + \dots c_{N-1}(t)V_{N-1} + V_{N+1} + V_{N+2} + \dots \\ &\subset c_1(r)V_1 + \dots c_{N-1}(t)V_{N-1} + V_{N+1} + V_{N+1},\end{aligned}$$

$$\varphi(s) \subset c_1(s)V_1 + \dots c_{N-1}(s)V_{N-1} + V_{N+1} + V_{N+1},$$

and

$$\begin{aligned}\varphi(r)+\varphi(s) &\subset c_1(r+s)V_1 + \dots c_{N-1}(r+s)V_{N-1} + V_{N+1} + V_{N+1} + V_{N+1} + V_{N+1} \\ &\subset c_1(r+s)V_1 + \dots c_{N-1}(r+s)V_{N-1} + V_N \subset \varphi(r+s).\end{aligned}$$

■

Lemma 3.23 For all $r \in D \cup [1,\infty)$:

$$0 \in \varphi(r).$$

Proof. For every r, $\varphi(r)$ is non-empty and it contains a neighborhood of zero. ■

Lemma 3.24 The set

$$\{\varphi(r) : r \in D\}$$

is totally ordered:

$$r < t \quad \text{implies} \quad \varphi(r) \subset \varphi(t).$$

Proof. By the first lemma, for $r < t$:

$$\varphi(r) \subset \varphi(r) + \varphi(t-r) \subset \varphi(t).$$

■

Lemma 3.25 For all $x,y \in \mathscr{X}$:

$$f(x+y) \le f(x) + f(y).$$

Proof. Let $x,y \in \mathscr{X}$ be given. Note that the range of f is $[0,1]$, hence we can limit ourselves to the case where the right hand side is less than 1. Fix $\varepsilon > 0$. There are $r,s \in D$ such that

$$f(x) < r \qquad f(y) < s \qquad \text{and} \qquad r+s < f(x) + f(y) + \varepsilon.$$

Because $\{\varphi(r)\}$ is fully ordered, this implies that $x \in \varphi(r)$, $y \in \varphi(s)$, hence

$$x+y \in \varphi(r) + \varphi(s) \subset \varphi(r+s).$$

Thus,

$$f(x+y) \le r+s < f(x) + f(y) + \varepsilon,$$

which holds for every $\varepsilon > 0$. ■

Lemma 3.26 The function f satisfies the additional properties:

① $f(x) = f(-x)$.
② $f(0) = 0$.
③ $f(x) > 0$ for $x \ne 0$.

Proof. Since the $\varphi(r)$ are unions of balanced sets they are balanced, from which follows that $f(x) = f(-x)$. $f(0) = 0$ because $0 \in \varphi(r)$ for all $r \in D$. Finally, if $x \ne 0$, it does no belong to some V_n (by separation) i.e., to some $\varphi(s)$, and since the $\{\varphi(r)\}$ form an ordered set, it does not belong to $\varphi(r)$ for all $r < s$, from which follows that $f(x) > 0$. ■

Back to the main proof. The properties of f imply that $d(x,y) = f(x-y)$ is a metric on $\mathscr{X}$. It is symmetric, vanishes if and only if $x = y$, it is translationally invariant, and

$$\begin{aligned} d(x,y) &= f(x-y) = f(x-z-(y-z)) \\ &\le f(x-z) + f(y-z) \\ &= d(x,z) + d(z,y). \end{aligned}$$

We next want to show that this metric is compatible with the topology. Consider the d-open balls,

$$\mathfrak{B}(0,\delta) = \{x : d(x,0) < \delta\} = \{x : f(x) < \delta\} = \bigcup_{r<\delta} \varphi(r).$$

(We used the fact that $f(x) < t$ implies that $x \in \varphi(t)$.) In particular, if $\delta < 1/2^n$, then $\mathfrak{B}(0,\delta) \subset V_n$, which proves that the open balls, $\mathfrak{B}(0,1/2^n)$, form a local base.

The open balls are balanced because each $\varphi(r)$ is balanced and the union of balanced set is balanced. If the V_n are convex then $\varphi(r)$ are convex.

■

3.6 Cauchy sequences

Cauchy sequences in metric spaces

We know from second year undergrad: a sequence (x_n) in a metric space $(\mathscr{X},d)$ is called a Cauchy sequence if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{such that} \quad \forall m,n > N \qquad d(x_m,x_n) < \varepsilon.$$

Cauchy sequences in topological vector spaces

In a topological vector space we can define a Cauchy sequence independently of a metric. Take a local base $\mathscr{B}$. A sequence (x_n) is a Cauchy sequence if

$$\forall V \in \mathscr{B} \quad \exists N \in \mathbb{N} \quad \text{such that} \quad \forall m,n > N \qquad x_m - x_n \in V.$$

This definition is independent of the local base since if this is true for one local base it is true for any other local base.

Cauchy sequences in topological vector spaces with compatible metric

Is a d-Cauchy sequence the same as a τ-Cauchy sequence? Yes. In a topological vector space with a compatible metric the open metric balls are a local base.

The following proposition refers to a distance increasing map between two metric spaces:

Proposition 3.27 Let $(\mathscr{X},d_1)$ and $(\mathscr{Y},d_2)$ be metric spaces, where $(\mathscr{X},d_1)$ is complete. Let $E \subset \mathscr{X}_1$ be a closed set and suppose that

$$f : E \to \mathscr{Y}$$

is continuous and satisfies

$$d_2(f(x),f(x')) \geq d_1(x,x').$$

Then $f(E)$ is closed in $\mathscr{Y}$.

Proof. Take $y \in \overline{f(E)}$. Then there exists a sequence $f(x_n) \in f(E)$ such that $f(x_n) \to y$ in $\mathscr{Y}$. It follows that (x_n) is a Cauchy sequence, and since E is closed, the limit x is in E. By the continuity of f,

$$y = \lim_{n\to\infty} f(x_n) = f(x) \in f(E).$$

■

We found that finite-dimensional subspaces of topological vector spaces are closed. The following proposition provides a criterion for an infinite-dimensional subspace of a topological vector space to be closed.

Proposition 3.28 Let $\mathscr{X}$ be a topological vector space and let $\mathscr{Y} \subset \mathscr{X}$ be a linear subspace endowed with the induced topology. If $\mathscr{Y}$ is an F-space then it is closed in $\mathscr{X}$.

Proof. Recall that being an F-space means that the topology on $\mathscr{Y}$ is consistent with a complete translationally invariant metric.

Let d be such an invariant metric, and set

$$B_{1/n} = \{y \in \mathscr{Y} : d(y,0) < 1/n\},$$

which are open balls in $\mathscr{Y}$.

Let $U_n \in \mathscr{N}_0$ be neighborhoods of zero in $\mathscr{X}$ such that $U_n \cap \mathscr{Y} = B_{1/n}$ (such neighborhoods exist because the open sets in $\mathscr{Y}$ are intersections of open sets in $\mathscr{X}$ with $\mathscr{Y}$). Let then $V_n \in \mathscr{N}_0^{\text{sym}}$ such that $V_{n+1} \subset V_n$ and $V_n + V_n \subset U_n$.

Let $x \in \overline{\mathscr{Y}}$ (the closure of $\mathscr{Y}$ in $\mathscr{X}$). Define

$$E_n = (x + V_n) \cap \mathscr{Y}$$

(this intersection is not empty because every neighborhood of x intersects $\mathscr{Y}$). Suppose that $y_1, y_2 \in E_n$, then

$$y_1 - y_2 \in \mathscr{Y} \qquad \text{and} \qquad y_1 - y_2 \in V_n + V_n \subset U_n \qquad \Longrightarrow \qquad y_1 - y_2 \in B_{1/n}.$$

We now use the fact that $\mathscr{Y}$ is an F-space: The sets E_n are non-empty and their diameter tends to zero. Since $\mathscr{Y}$ is complete, the intersection of the $\mathscr{Y}$-closure of the sets E_n contains exactly one point, which we denote by y_0. Specifically, y_0 is the only point, such that for every $U' \in \mathscr{N}_0$ in $\mathscr{Y}$ and every n,

$$(y_0 + U') \cap [(x + V_n) \cap \mathscr{Y}] \neq \varnothing.$$

Since open neighborhoods in $\mathscr{Y}$ are intersections of open neighborhoods in $\mathscr{X}$ with $\mathscr{Y}$, it follows that for every $U \in \mathscr{N}_0$ in $\mathscr{X}$ and every n

$$(y_0 + U) \cap (x + V_n) \cap \mathscr{Y} \neq \varnothing,$$

Take now any neighborhood $W \in \mathcal{N}_0$ in $\mathcal{X}$ and define

$$F_n^W = (x + W \cap V_n) \cap \mathcal{Y}.$$

By the exact same same argument the intersection of the $\mathcal{Y}$ closure of the sets F_n^W contains exactly one point. But since $F_n^W \subset E_n$, this point has to be y_0. Thus, there exists a unique point y_0, such that for every $U, W \in \mathcal{N}_0$ in $\mathcal{X}$ and every n,

$$(y_0 + U) \cap (x + W \cap V_n) \cap \mathcal{Y} \neq \varnothing.$$

Since $x + W \cap V_n \subset x + W$, it follows that for every $U, W \in \mathcal{N}_0$

$$(y_0 + U) \cap (x + W) \neq \varnothing.$$

Since the space is Hausdorff, $x = y_0 \in \mathcal{Y}$, i.e., $\overline{\mathcal{Y}} = \mathcal{Y}$. ■

The following lemma will turn to be useful:

Lemma 3.29

① Let d be a translation invariant metric on a vector space $\mathcal{X}$, then

$$d(nx, 0) \leq n\, d(x, 0).$$

② If $x_n \to 0$ in a metrizable topological vector space, then there exist positive scalars $\alpha_n \to \infty$, such that $\alpha_n x_n \to 0$.

Proof. The first part is obvious by successive applications of the triangle inequality,

$$d(nx, 0) \leq \sum_{k=1}^{n} d(kx, (k-1)x) \leq n\, d(x, 0).$$

For the second, we note that since $d(x_n, 0) \to 0$, there exists a diverging sequence of positive integers n_k, such that

$$d(x_k, 0) \leq \frac{1}{n_k^2},$$

from which we get that

$$d(n_k x_k, 0) \leq n_k\, d(x_k, 0) \leq \frac{1}{n_k} \to 0.$$

■

3.7 Boundedness and continuity

We defined bounded sets in topological vector spaces. When the space is metrizable, one could also define a set to be bounded if its diameter is finite. The two definitions are not equivalent. For example, the metric that was defined in Theorem 3.21 satisfies $d(x,y) \le 1$, so that even $\mathscr{X}$ is a bounded set. We will always refer to the topological notion of boundedness.

We already know that compact sets are bounded (Theorem 3.12), and that the closure of a bounded set is bounded (Proposition 3.8).

Here is a simple exercise:

Proposition 3.30 A Cauchy sequence (and in particular a converging sequence) in a topological vector space is bounded.

Proof. Let (x_n) be a Cauchy sequence. Let $W \in \mathscr{N}_0^{\text{bal}}$, and let $V \in \mathscr{N}_0^{\text{bal}}$ satisfy

$$V + V \subset W.$$

By the definition of a Cauchy sequence, there exists an N such that for all $m, n \ge N$,

$$x_n - x_m \in V,$$

and in particular,

$$\forall n > N \qquad x_n \in x_N + V.$$

Set $s > 1$ such that $x_N \in sV$ (we know that such an S exists), then for all $n > N$,

$$x_n \in sV + V \subset sV + sV \subset sW.$$

Since for balanced sets $sW \subset tW$ for $s < t$, and since every neighborhood of zero contains a balanced neighborhood, this proves that the sequence is indeed bounded. ∎

Proposition 3.31 For every $x \ne 0$ the set $E = \{nx : n \in \mathbb{N}\}$ is not bounded.

Proof. By separation, there exists a neighborhood $V \in \mathscr{N}_0$ that does not contain x, hence $nx \notin nV$, i.e., for every n,

$$E \not\subset nV.$$

∎

Corollary 3.32 The only bounded subspace of a topological vector space is $\{0\}$.

Proposition 3.33 Let $\mathscr{X}$ be a topological vector space and let $E \subset \mathscr{X}$. Then, E is bounded if and only if for every sequence $x_n \in E$ and every sequence of scalars $\alpha_n \to 0$, $\alpha_n x_n \to 0$.

Proof. **Suppose that E is bounded**. Let $\alpha_n \to 0$. Let $V \in \mathscr{N}_0^{\text{bal}}$. Then, $E \subset tV$ for some t, and for sufficiently large n, $\alpha_n t < 1$, in which case

$$\alpha_n E \subset (\alpha_n t)V \subset V.$$

Thus, for every sequence $x_n \in E$, $\alpha_n x_n$ is eventually in any neighborhood of zero.

Suppose that for every sequence $x_n \in E$ and every $\alpha_n \to 0$, $\alpha_n x_n \to 0$. Suppose, by contradiction, that E were not bounded. Then there exists a $V \in \mathscr{N}_0$ and a sequence $r_n \to \infty$, such that no $r_n V$ contains E. Take then a sequence $x_n \in E$ such that $x_n \notin r_n V$. Thus,

$$r_n^{-1} x_n \notin V,$$

which implies that $r_n^{-1} x_n \not\to 0$, which is a contradiction. ■

We are now in measure to establish the relations between continuous and bounded linear transformations. As we shall see, the relation is more involved than in normed spaces, where the two notions coincide.

Theorem 3.34 Let $T : \mathscr{X} \to \mathscr{Y}$ be a linear map between topological vector spaces. Consider the following four properties:

① T is continuous.
② T is bounded.
③ If $x_n \to 0$ then $\{T(x_n) \mid n \in \mathbb{N}\}$ is bounded.
④ If $x_n \to 0$ then $T(x_n) \to 0$.

Then,

$$① \implies ② \implies ③.$$

If, moreover, $\mathscr{X}$ is metrizable, then

$$③ \implies ④ \implies ①,$$

i.e., all four are equivalent.

Proof. ①→②. Suppose that T is continuous. Let $V \subset \mathscr{X}$ be bounded. Let $W \in \mathscr{N}_0$ in $\mathscr{Y}$. Since $T(0) = 0$,

$$\exists U \in \mathscr{N}_0 \qquad T(U) \in W.$$

Because V is bounded there exists an s such that for all $t > s$,

$$V \subset tU.$$

By linearity, for all $t > s$,

$$T(V) \subset T(tU) = t\,T(U) \subset tW,$$

hence $T(V)$ is bounded, which implies that T is bounded.

②→③. Suppose that T is bounded and let $x_n \to 0$. Since convergent sequences are bounded (Proposition 3.30), namely,

$$E = \{x_n \mid n \in \mathbb{N}\}$$

is bounded, then

$$T(E) = \{T(x_n) \mid n \in \mathbb{N}\}$$

is bounded.

③→④. Suppose now that $\mathscr{X}$ is metrizable and let d be a compatible and invariant metric. Suppose that $x_n \to 0$ and $\{T(x_n) \mid n \in \mathbb{N}\}$ is bounded. By the second part of Lemma 3.29 (and here we need metrizability), there exists a sequence $\alpha_n \to \infty$, such that $\alpha_n x_n \to 0$. Then,

$$T(x_n) = \frac{1}{\alpha_n} \underbrace{T(\alpha_n x_n)}_{\text{bounded}},$$

and by Proposition 3.33, $T(x_n) \to 0$.

④→①. For any first countable topological space, sequential continuity implies continuity. Any metrizable topological vector space is first countable (has a countable local base) because the open balls are bounded. ■

3.8 Seminorms and local convexity

Definition 3.17 A **seminorm** (סמי נורמה) on a vector space $\mathscr{X}$ is function $p : \mathscr{X} \to \mathbb{R}$, such that

$$p(x+y) \le p(x) + p(x),$$

and

$$p(\alpha x) = |\alpha|\, p(x).$$

Definition 3.18 Let $\mathscr{P}$ be a family of seminorms. It is called **separating** if to each $x \neq 0$ corresponds a $p \in \mathscr{P}$, such that $p(x) \neq 0$.

Comment 3.5 The only difference between a norm and a seminorm is that the latter does not separate between points; knowing that $p(y-x) = 0$ does not guarantee that $x = y$. While a single seminorm does not separate between points, a family of seminorms might do so.

Recall the definition of **absorbing sets** (**קבוצה בולעת**). In fact, we proved that every neighborhood of zero is absorbing. For convex absorbing sets K (in a vector space—no topology required) we defined the **Minkowski functionals**,

$$p_K(x) = \inf\{t > 0 : x/t \in K\}.$$

We proved that the Minkowski functionals are sub-additive, and $p_K(\alpha x) = \alpha\, p_K(x)$ for $\alpha > 0$. Also, $p_K(x) \leq 1$ for $x \in K$ and $p_K(x) < 1$ for internal points (**נקודות תוך**) of K.

There turns out to be a strong relation between Minkowski functionals and seminorms: every seminorm is a Minkowski functional of some balanced, convex, absorbing set. This follows from the following propositions:

Proposition 3.35 Let p be a seminorm on a vector space $\mathscr{X}$ (no topology). Then:

① p is symmetric.
② $p(0) = 0$.
③ $|p(x) - p(y)| \leq p(x-y)$.
④ $p(x) \geq 0$.
⑤ $\ker p$ is a linear subspace.
⑥ The set $B = \{x : p(x) < 1\}$ is convex, balanced and absorbing.
⑦ $p = p_B$.

Proof. By the properties of the seminorm:

① $p(x-y) = p(-(y-x)) = |-1|\,p(y-x) = p(y-x)$.
② $p(0) = p(0 \cdot x) = 0 \cdot p(x) = 0$.
③ This follows from the inequalities

$$p(x) \leq p(y) + p(x-y) \qquad \text{and} \qquad p(y) \leq p(x) + p(y-x) = p(x) + p(x-y).$$

④ By the previous item, for every x:

$$0 \leq |p(x) - p(0)| \leq p(x).$$

⑤ If $x, y \in \ker p$:

$$p(\alpha x + \beta y) \le p(\alpha x) + p(\beta y) = |\alpha|\, p(x) + |\beta|\, p(y) = 0.$$

⑥ If $p(x) < 1$ then for every $|\alpha| \le 1$, $p(\alpha x) = |\alpha| p(x) < 1$, which proves that B is balanced. If $x, y \in B$ then for every $0 \le t \le 1$

$$p(tx + (1-t)y) \le t\, p(x) + (1-t)\, p(y) < 1,$$

hence B is convex. For every $x \in \mathscr{X}$ and $s > p(x)$, $p(x/s) < 1$, i.e., $x \in sB$, so that B is absorbing.

⑦ Finally, consider

$$p_B(x) = \inf\{s > 0 : x/s \in B\} = \inf\{s > 0 : p(x/s) < 1\} = \inf\{s > 0 : p(x) < s\}.$$

If $p(x) < r$ then $p_B(x) < r$, i.e., $p_B \le p$. If $p_B(x) < r$, then there exists an $s < p_B(x)$ such that $p(x) = s$, i.e., $p \le p_B$.

■

And now for the reverse direction, which we have already shown:

Proposition 3.36 Let K be a convex absorbing set in a vector space $\mathscr{X}$. Then

① $p_K(x+y) \le p_K(x) + p_K(y)$.
② $p_K(tx) = t\, p_K(x)$ for all $t > 0$.
③ If K is also balanced then $p_K(\alpha x) = |\alpha|\, p_K(x)$.

To conclude:

Corollary 3.37 In a vector space $\mathscr{X}$ there is a one-to-one correspondence between the seminorms and Minkowski functionals on convex balanced absorbing sets.

Up to now we were in the realm of vector space with no topology. We proceed to see the importance of seminorms in the context of topological vector spaces. Recall that in a topological vector space every open neighborhood of zero is absorbing and contains a balanced neighborhood. What we need in addition is convexity.

Proposition 3.38 Let $\mathscr{B}$ be a convex and balanced local base in a (locally convex!) topological vector space $\mathscr{X}$. Then,

① $\forall V \in \mathscr{B} \qquad V = \{x : p_V(x) < 1\}$.

② $\{p_V \mid V \in \mathscr{B}\}$ is a separating family of continuous seminorms.

Comment 3.6 This is essentially what local convexity gives—the fact that the Minkowski functional over that base are both continuous and separating.

Proof.

① Let $x \in V$. Since V is open, there is a $t < 1$ such that $x/t \in V$, which implies that

$$p_V(x) = \inf\{s > 0 : x/s \in V\} < 1$$

i.e.,

$$V \subset \{x \mid p_V(x) < 1\}.$$

Conversely, if $x \notin V$, then by convexity $x/t \in V$ only for $t > 1$, i.e., $p_V(x) \geq 1$.

② We already know that the functions p_V are seminorms. We will first show that this family of seminorms is separating. Let $x \neq 0$. Let $V \in \mathscr{B}$ be such that $x \notin V$ (separation). Then $p_V(x) \geq 1 > 0$. The continuity of the p_V's follows from the inequality

$$|p_V(x) - p_V(0)| \leq p_V(x).$$

Recall that $\{V/n : n \in \mathbb{N}\}$ is a local base. If $x \in V/n$ then

$$|p_V(x) - p_V(0)| \leq p_V(x) = \frac{1}{n} p_V(nx) < \frac{1}{n}.$$

That is, for all $\varepsilon > 0$ there exists a neighborhood $U \in \mathscr{N}_0$ such that

$$p_V(U) \subset [0, \varepsilon).$$

■

And why do we care about separating families of seminorms? The answer is below: a separating family of seminorms allows us to define a topology on a vector space. In fact, very often a vector space is given without any topology, and the way to topologize it by means of a separating family of seminorms.

Theorem 3.39 Let $\mathscr{P}$ be a separating family of seminorms on a vector space $\mathscr{X}$. For each $p \in \mathscr{P}$ and $n \in \mathbb{N}$ define

$$V(p,n) = \{x : p(x) < 1/n\}.$$

Let $\mathscr{B}$ be the set of all finite intersections of $V(p,n)$'s. Then, $\mathscr{B}$ is a convex balanced local base for a topology τ on $\mathscr{X}$ (that is, $(\mathscr{X}, \tau)$ is locally convex with respect to this topology). Moreover,

① Every $p \in \mathscr{P}$ is continuous.

② A set $E \subset \mathscr{X}$ is bounded if and only if p is bounded on E for every $p \in \mathscr{P}$.

Proof. We defined

$$\mathscr{B} = \{ \bigcap_{(p,n)\in I} V(p,n) \mid I \subset \mathscr{P} \times \mathbb{N}, |I| < \infty \},$$

where all the intersections are finite. Clearly, $\mathscr{B}$ is a collection of sets that contain the origin and that are closed under finite intersections.

Set $A \subset \mathscr{X}$ be be open if it is a union of translates of members of B. That is, A is open if and only if it can be represented in the form

$$A = \bigcup_{x\in\mathscr{X}, B(x)\in\mathscr{B}} (x + B(x)),$$

where the union can be empty. We denote the collection of all such set by τ. τ is a translation-invariant topology on $\mathscr{X}$ because:

① $\varnothing \in \tau$.
② $\mathscr{X} \in \tau$ (obvious as well).
③ τ is closed under arbitrary unions.
④ τ is closed under finite intersection as

$$\bigcup_{x\in I, B(x)\in\mathscr{B}_I} (x + B(x)) \cap \bigcup_{y\in J, B(y)\in\mathscr{B}_J} (y + B(y)) = \bigcup_{x\in I, B(x)\in\mathscr{B}_I} \bigcup_{y\in J, B(y)\in\mathscr{B}_J} (x + B(x)) \cup (y + B(y))$$
$$= ????$$

We know that each $V(p,n)$ is convex and balanced. An intersection of convex balanced sets is convex and balanced, hence $\mathscr{B}$ constitutes a convex balanced local base.

We need to show that the vector space $\mathscr{X}$ with the topology τ is a topological vector space. Let $0 \neq x \in \mathscr{X}$. Since the family of seminorms is separating,

$$\exists p \in \mathscr{P} \qquad p(x) > 0.$$

For $n \in \mathbb{N}$ such that $n\,p(x) \geq 1$,

$$p(x) = \frac{n\,p(x)}{n} \geq \frac{1}{n},$$

i.e., $x \notin V(p,n)$, or $0 \notin x - V(p,n)$. This means that the complement of $\{0\}$ is open, i.e., $\{0\}$ is closed. By translation-invariance, every $\{x\}$ is closed.

We next show that addition is continuous. Let $U \in \mathscr{N}_0$. Then, by the property of the local base, there exist

$$V(p_1,n_1) \cap \cdots \cap V(p_k,n_k) \subset U.$$

Let

$$W = V(p_1,2n_1) \cap \cdots \cap V(p_k,2n_k),$$

then since seminorms are subadditive,

$$\begin{aligned} V(p_1,2n_1) + V(p_1,2n_1) &= \{x \mid p_1(x) < 1/2n\} + \{x \mid p_1(x) < 1/2n\} \\ &= \{x+y \mid p_1(x) < 1/2n, p_1(y) < 1/2n\} \\ &\subset \{x+y \mid p_1(x+y) < 1/n\} = V(p_1,n_1), \end{aligned}$$

from which we conclude that

$$W + W \subset U.$$

This proves that addition is continuous! A similar argument shows that scalar multiplication is continuous as well.

$\mathscr{X}$ is a locally convex topological vector space. Every $p \in \mathscr{P}$ is continuous at 0 because for every $\varepsilon > 0$ set $n > 1/\varepsilon$ and

$$p(V(p,n)) \subset B(0,\varepsilon).$$

It is continuous everywhere by

$$|p(x) - p(y)| \le p(x-y).$$

Finally, suppose that E is bounded. Take $p \in \mathscr{P}$. Then

$$V(p,1) = \{x \in \mathscr{X} : p(x) < 1\}$$

is a neighborhood of zero. Hence,

$$E \subset nV(p,1)$$

for some n (by definition of boundedness). Hence, for all $x \in E$,

$$x \in \{nx \in \mathscr{X} : p(x) < 1\} = \{nx \in \mathscr{X} : p(nx) < n\} = \{y \in \mathscr{X} : p(y) < n\},$$

i.e., $p(x) < n$.

Conversely, if $p(E)$ is bounded for every $p \in \mathscr{P}$. Then there are numbers M_p such that

$$\sup_{x \in E} p(x) < M_p.$$

Let $U \in \mathscr{N}_0$. Again, there are

$$V(p_1,n_1) \cap \cdots \cap V(p_k,n_k) \subset U.$$

For sufficiently large n,

$$E \subset n(V(p_1,n_1) \cap \cdots \cap V(p_k,n_k)) \subset nU,$$

i.e., E is bounded. ■

Comments 3.1

① Let (X,τ) be a locally convex topological vector space, and let $\mathscr{B}$ be a convex balanced local base. This local base defines a separating family of seminorms on $\mathscr{X}$, which by the previous theorem induces a topology τ' on $\mathscr{X}$. Is this topology the same as τ? The answer is positive.

② If $\mathscr{P} = (p_n)$ is countable separating family of seminorms, then the resulting topology has a countable local base, and by Theorem 3.21 it is metrizable. One can show that the following is a compatible metric:

$$d(x,y) = \max_n \frac{c_n\, p_n(x-y)}{1+p_n(x-y)},$$

where (c_n) is any sequence of positive numbers that decays to zero (it is easy to see that the maximum is indeed attained). Clearly, $d(x,x) = 0$. Also, since the p_n's are separating $d(x,y) > 0$ for $x \neq y$. Symmetry, as well as translational invariance are obvious. Finally, the triangle inequality follows form the fact that every p_n is subadditive, and that $a \le b+c$ implies that

$$\frac{a}{1+a} \le \frac{b}{1+b} + \frac{c}{1+c}.$$

It remains to show that this metric is compatible with the topology.

And finally:

Theorem 3.40 $(\mathscr{X},\tau)$ is normable if and only if there exists a convex bounded neighborhood.

Proof. If $\mathscr{X}$ is normable then $\mathfrak{B}_{\mathscr{X}} = \{x : \|x\| < 1\}$ is convex and bounded.

Suppose that there exists a convex and bounded $V \in \mathscr{N}_0$. Then there exists, by Theorem 3.9 a convex and balanced (and certainly bounded) neighborhood $U \subset V$. Set

$$\|x\| = p_U(x).$$

We will show that this is indeed a norm. Clearly, $\|x\| = 0$ if and only if $x = 0$. Since U is balanced then $p_U(\alpha x) = |\alpha|\, p_U(x)$. The triangle inequality follows from the properties of p_U. It remains to show that this norm is compatible with the topology. This follows from the fact that

$$\mathfrak{B}(0,r) = \{x : \|x\| < r\} = \{x : p_U(x) < r\} = \{x : p_U(x/r) < 1\} \subset rU,$$

which means that $\mathfrak{B}(0,r)$ is bounded, hence

$$\{\mathfrak{B}(0,r) : r > 0\}$$

is a local base. ∎

3.9 Examples

3.9.1 The space $C(\Omega)$

Let Ω an open set in $\mathbb{R}^n$. We consider the space $C(\Omega)$ of all continuous functions. Note that the sup-norm does not work here. There exist unbounded continuous functions on open sets.

Every open set Ω in $\mathbb{R}^n$ can be written as

$$\Omega = \bigcup_{n=1}^{\infty} K_n,$$

where $K_n \Subset K_{n+1}$, where the K_n are compact, and $\Subset$ stands for **compactly embedded**, i.e., K_n is a compact set in the interior of K_{n+1}. We topologize $C(\Omega)$ with the separating family of seminorms,

$$p_n(f) = \sup\{|f(x)| : x \in K_n\} \equiv \|f\|_{K_n}.$$

(These are clearly seminorms, and they are separating because for every $f \neq 0$ there exists an n such that $f|_{K_n} \neq 0$.)

Since the p_n's are monotonically increasing,

$$\bigcap_{m=1}^{M} \bigcap_{k=1}^{n} V(p_k, m) = \bigcap_{m=1}^{M} \bigcap_{k=1}^{n} \{f : p_k(f) < 1/m\} = V(p_n, M),$$

which means that the $V(p_n, M)$ form a convex local base for $C(\Omega)$. In fact, $V(p_n, M)$ contains a neighborhood obtained by taking n, M to be the greatest of the two, from which follows that

$$V_n = \{f : p_n(f) < 1/n\}$$

is a convex local base for $C(\Omega)$, and the p_n's are continuous in this topology.

We can thus endow this topological space with a compatible metric, for example,

$$d(f, g) = \max_n \frac{2^{-n} p_n(f-g)}{1 + p_n(f-g)}.$$

We will now show that this space is complete. Recall that if a topological vector space has a compatible metric with respect to which it is complete, then it is called an F-space. If, moreover, the space is locally convex, then it is called a Fréchet space. Thus, $C(\Omega)$ is a Fréchet space.

Let then (f_n) be a Cauchy sequence. This means that for every $\varepsilon > 0$ there exists an N, such that for every $m, n > N$,

$$\max_k \frac{2^{-k} p_k(f_n - f_m)}{1 + p_k(f_n - f_m)} < \varepsilon,$$

and a fortiori,

$$(\forall k \in \mathbb{N}) \qquad \frac{2^{-k} p_k(f_n - f_m)}{1 + p_k(f_n - f_m)} < \varepsilon,$$

which means that (f_n) is a Cauchy sequence in each K_k (endowed with the sup-norm), and hence converges uniformly to a function f.

Given ε let K be such $2^{-K} < \varepsilon$, then

$$\max_{k>K} \frac{2^{-k} p_k(f_n - f)}{1 + p_k(f_n - f)} < \varepsilon,$$

and there exists an N, such that for every $n > N$,

$$\max_{k\le K} \frac{2^{-k} p_k(f_n - f)}{1 + p_k(f_n - f)} < \varepsilon,$$

which implies that $f_n \to f$, hence the space is indeed complete.

Remains the question whether $C(\Omega)$ with this topology is normable. For this, the origin must have a convex bounded neighborhood. Recall that a set E is bounded if and only if $\{p_n(f) : f \in E\}$ is bounded for every n, i.e., if

$$\{\sup\{|f(x)| : x \in K_n\} : f \in E\}$$

is a bounded set for every n, or if

$$\forall n \in \mathbb{N} \qquad \sup\{|f(x)| : x \in K_n, f \in E\} < \infty.$$

Because the V_n form a base, every neighborhood of zero contains a set

$$V_k = \{f : p_k(f) < 1/k\},$$

hence,

$$\sup\{|f(x)| : x \in K_n, f \in E\} \ge \sup\{\|f\|_{K_n} : \|f\|_{K_k} < 1/k\}.$$

The right hand side can be made as large as we please for $n > k$, i.e., no set is bounded, and hence the space is not normable.

3.9.2 The space $C^\infty(\mathbb{R})$

For simplicity, we consider in the section functions $\mathbb{R} \to \mathbb{R}$.

> **Definition 3.19** A function $f : \mathbb{R} \to \mathbb{R}$ belongs to the space $C^\infty(\mathbb{R})$ if for every $k \in \mathbb{N}$, $f^{(k)} \in C$.

Definition 3.20 The **support** (תומך) of a function $f:\mathbb{R}\to\mathbb{R}$ is

$$\operatorname{Supp} f = \overline{\{x : f(x) \neq 0\}}.$$

Definition 3.21 Let $K \subset \mathbb{R}$ be compact. The space $\mathscr{D}_K$ is the subspace of $C^\infty(\mathbb{R})$-functions whose support lies in K.

We endow $C^\infty(\mathbb{R})$ with a topology, turning it into a Fréchet space, such that for every compact set K, $\mathscr{D}_K$ is a closed subspace. As above, let (K_n) be an increasing sequence of compact sets covering $\mathbb{R}$. We define the seminorms:

$$p_n(f) = \max\{|f^{(k)}(x)| : x \in K_n, k \leq n\}.$$

These seminorms are a separating family, hence

Exercise 3.5 In undergraduate calculus, you learned to distinguish between pointwise convergence and uniform convergence. In particular, you learned that the latter is convergence in a metric space (in fact a normed space)—the space of functions on a compact set endowed with the sup-norm. Pointwise convergence, of the other hand, was not presented as a mode of convergence in any topological space.

Consider then the space $\mathscr{X}$ of functions $[0,1]\to\mathbb{R}$. We topologize it with the separating family of seminorms:

$$\mathscr{P} = \{p_x : x \in [0,1]\} \qquad p_x(f) = |f(x)|.$$

① Show that this is indeed a separating family of seminorms, and describe the induced topology τ.
② Show that there exists a sequence $f_n \to 0$, such that for every sequence $\alpha_n \to \infty$, $\alpha_n f_n \not\to 0$.
③ Conclude that $(\mathscr{X}, \tau)$ is not metrizable.

■

3.10 Completeness

This section is devoted to the notion of completeness. Most of the content of the second chapter of Rudin coincides with material learned along this course. We will only highlight differences that result from the more general context.

3.10.1 The Banach-Steinhaus theorem

Recall the Banach-Steinhaus theorem, as we proved it for Banach spaces:

Theorem 3.41 Let $\mathscr{X}$ be a Banach space and let $\mathscr{Y}$ be a normed space. Let $\{T_\alpha : \alpha \in A\}$ be a collection of bounded linear maps, such that

$$(\forall x \in \mathscr{X}) \qquad \sup_\alpha \|Tx\| < \infty.$$

Then,

$$\sup_\alpha \|T_\alpha\| < \infty.$$

We will now see how this generalizes for topological vector spaces. First, a definition:

Definition 3.22 Let $\mathscr{X}, \mathscr{Y}$ be topological vector spaces. let $\{T_\alpha : \alpha \in A\}$ be a collection of linear maps $\mathscr{X} \to \mathscr{Y}$. This collection is **equicontinuous** (**רציפה במידה אחידה**) if

$$(\forall U \in \mathscr{N}_0^{\mathscr{Y}})(\exists V \in \mathscr{N}_0^{\mathscr{X}}) : (\forall \alpha \in A)(T_\alpha(V) \subset U).$$

Comment 3.7 Compare with the definition of equicontinuity in a metric space.

We have shown that continuity implies boundedness. We will now show that equicontinuity implies uniform boundedness:

Proposition 3.42 Let $\mathscr{X}, \mathscr{Y}$ be topological vector spaces. Let $\{T_\alpha : \alpha \in A\}$ be a equicontinuous collection of linear maps $\mathscr{X} \to \mathscr{Y}$. Then to every bounded set $E \subset \mathscr{X}$ corresponds a bounded set $F \subset \mathscr{Y}$, such that

$$(\forall \alpha \in A) \qquad T_\alpha(E) \subset F.$$

We say then that $\{T_\alpha : \alpha \in A\}$ is uniformly bounded.

Proof. Let $E \subset \mathscr{X}$ be bounded, and set

$$F = \bigcup_{\alpha \in A} T_\alpha(E).$$

Obviously, $T_\alpha(E) \subset F$ for all $\alpha \in A$, so it only remains to prove that F is bounded. Let $U \in \mathscr{N}_0^{\mathscr{Y}}$. Because $\{T_\alpha\}$ is equicontinuous, there exists a $V \in \mathscr{N}_0^{\mathscr{X}}$ such that

$$(\forall \alpha \in A) \quad T_\alpha(V) \subset U.$$

Since E is bounded, there exists an $s > 0$ such that

$$(\forall t > s) \quad E \subset tV.$$

Then,

$$(\forall \alpha \in A)(\forall t > s) \quad T_\alpha(E) \subset T_\alpha(tV) = t\,T_\alpha(V) \subset tU,$$

hence $F \subset tU$, which proves that F is bounded. ■

This is the Banach-Steinhaus theorem in the context of topological vector spaces:

Theorem 3.43 Let $\mathscr{X}$ and $\mathscr{Y}$ be topological vector spaces. Let $\{T_\alpha : \alpha \in A\}$ be a collection of continuous linear mappings $\mathscr{X} \to \mathscr{Y}$. Define

$$B = \{x \in \mathscr{X} : \{T_\alpha(x) : \alpha \in A\} \text{ is bounded}\}.$$

If B is of the second category (i.e., not a countable union of nowhere dense sets), then $B = \mathscr{X}$ and the $\{T_\alpha\}$ are equicontinuous.

Comment 3.8 OK, let's remind ourselves what is this category stuff. A set is called **nowhere dense** (דליל) if its closure has an empty interior. An example of a nowhere dense set in $\mathbb{R}$ is the set

$$\{1/n : n \in \mathbb{N}\}.$$

The complement of the closure of a nowhere dense set is dense.

A set is called of the **first category** if it is a countable union of nowhere dense sets. An example of a set of the first category is the set of functions in $C[0,1]$ that are differentiable at some point. A set that is not of the first category is of the **second category**.

Baire's theorem states that in a complete metric set, if the space can be covered by a countable union of closed set, then one of them contains an open set. (Equivalently, Baire's theorem states that the intersection of every countable collection of dense open sets is dense.)

Let E_n be a countable collection of nowhere dense sets. Then, $(\overline{E_n})^c$ is open and dense, and by Baire's theorem,

$$\bigcap_{n=1}^{\infty} (\overline{E_n})^c \neq \varnothing.$$

Taking complements,

$$\bigcup_{n=1}^{\infty} (\overline{E_n}) \neq \mathscr{X}.$$

That is, a complete metric space is not a countable union of nowhere dense sets—it is of the second category.

Proof. Choose $U, W \in \mathscr{N}_0^{\text{bal}}$ in $\mathscr{Y}$, such that

$$\overline{U} + \overline{U} \subset W,$$

and set

$$E = \bigcap_{\alpha \in A} T_\alpha^{-1}(\overline{U}) = \{x \in \mathscr{X} : \forall \alpha \in A, T_\alpha(x) \in \overline{U}\}.$$

Since each of the T_α is continuous, and E is an intersection of closed sets, it is closed.

By definition, if $x \in B$, then the set $\{T_\alpha(x) : \alpha \in A\}$ is bounded in $\mathscr{Y}$. Thus,

$$(\exists n \in \mathbb{N}) : \{T_\alpha(x) : \alpha \in A\} \subset nU.$$

Thus $x \in nE$, namely.

$$B \subset \bigcup_{n=1}^{\infty} nE.$$

If B is of the second category, then at least one of the nE is of the second category[3], i.e., E itself is of the second category. A set of the second category has an interior point, say x, hence $E - x$ contains a neighborhood V of zero. Then for every $\alpha \in A$,

$$T_\alpha(V) \subset T_\alpha(x) - T_\alpha(E) \subset \overline{U} - \overline{U} \subset W,$$

which proves that $\{T_\alpha\}$ is equicontinuous.

It thus follows that $\{T_\alpha\}$ is uniformly bounded, i.e., for every x,

$$\{T_\alpha(x) : \alpha \in A\}$$

is bounded, i.e., $B = \mathscr{X}$. ■

Corollary 3.44 If $\{T_\alpha\}$ is a collection of continuous linear maps from an F-space $\mathscr{X}$ into a topological vector space $\mathscr{Y}$, and if the sets

$$\{T_\alpha(x) : \alpha \in A\}$$

are bounded (in $\mathscr{Y}$) for every $x \in \mathscr{X}$, then $\{T_\alpha\}$ is equicontinuous.

Proof. This is an immediate consequence of the previous theorem, because an F-space is of the second category. ■

[3] A countable union of sets of the first category is also of the first category.

How does this last corollary translate in the case of a Banach space? If $\mathscr{X}$ is a Banach space and $\mathscr{Y}$ is a normed space, then if

$$\sup_{\alpha\in A} \|T_\alpha x\| < \infty,$$

then the $\{T_\alpha\}$ are equicontinuous, which in turn implies that they are uniformly bounded. That is, for every there exists an M such that the (bounded) unit ball in $\mathscr{X}$ is mapped into a bounded set in $\mathscr{Y}$,

$$(\forall x \in \mathscr{X} : \|x\| \le 1) \qquad \|T_\alpha(x)\| \le M,$$

and hence,

$$\sup_{\alpha\in A} \|T_\alpha\| < \infty,$$

which is our good old Banach-Steinhaus theorem.

3.11 The Hahn-Banach theorem

We felt with the Hahn-Banach theorem in Chapter 2, but recall that most of the content had nothing to do with Banach spaces. Let us first refresh our memory.

The theorem that we called the Hahn-Banach theorem was the following:

Theorem 3.45 — Hahn-Banach. Let $\mathscr{V}$ be a real vector space and let p a functional over $\mathscr{V}$ that satisfies:

① Sub-linearity: $p(x+y) \le p(x) + p(y)$.
② Homogeneity: for every $\alpha \ge 0$, $p(\alpha x) = \alpha p(x)$.

Let $\mathscr{Y} \subset \mathscr{V}$ be a linear subspace and let $f : \mathscr{Y} \to \mathbb{R}$ be a linear functional satisfying:

$$\langle f, \cdot\rangle \le p|_{\mathscr{Y}}.$$

There there exists a linear functional F on $\mathscr{V}$ such that $F|_{\mathscr{Y}} = f$ and $\langle F, \cdot\rangle \le p$.

Note that the Hahn-Banach involves no topology. It is a theorem about a vector space, a linear subspace, and linear and nonlinear functionals. Also, note that any seminorm (a term that we did not used back then) satisfies the requirements on p.

Then came an extension theorem:

Theorem 3.46 — Extension theorem. Let $\mathscr{X}$ be a normed space and let $\mathscr{Y}$ be a linear subspace. Let $f \in \mathscr{Y}^*$. Then there exists an $F \in \mathscr{X}^*$, such that $F|_{\mathscr{Y}} = f$

and $\|F\| = \|f\|$.

Obviously, this theorem does not apply to topological vector spaces.

Definition 3.23 Let $\mathscr{X}$ be a topological vector space. Its dual, $\mathscr{X}^*$, is the vector space of continuous linear functionals on $\mathscr{X}$.

Comment 3.9 As for now, the dual space is only a vector space, but not a topological vector space.

We also had a separation theorem:

Theorem 3.47 — Separation theorem. Let M and N be disjoint convex sets in a vector space $\mathscr{V}$. If at least one of them, say M, has an internal point, then there exists a non-zero linear functional that separates between M and N.

Note that this theorem refers to an internal point, which unlike an interior point is not a topological concept; it is a vector space concept.

In contrast, the following corollary applies to normed spaces:

Corollary 3.48 Let $\mathscr{X}$ be a normed space and let K_1, K_2 be disjoint convex sets that at least one of them has an interior point. Then there exists a bounded linear functional $f \neq 0$ that separates K_1 and K_2.

Yes, this theorem does carry on to topological vector spaces.

A very important consequence of the Hahn-Banach theorem is the following:

Proposition 3.49 In a locally-convex topological vector space $\mathscr{X}$, $\mathscr{X}^*$ separates points of $\mathscr{X}$.

3.12 Weak topologies

3.12.1 General statements

Definition 3.24 Let S be a set and let τ_1 and τ_2 be topologies on S. τ_1 is said to be **weaker** than τ_2 if

$$\tau_1 \subset \tau_2.$$

This means that every τ_1-open set is a τ_2-open set.

Comment 3.10 A topology τ_1 is weaker than a topology τ_2 if for every τ_1-open set A, every point $a \in A$ has a τ_2-neighborhood U_a contained in A, as

$$A = \bigcup_{a \in A} U_a,$$

i.e., A is also τ_2-open.

Comment 3.11 If τ_1 is weaker than τ_2 then

$$\mathrm{Id} : (S, \tau_1) \to (S, \tau_2)$$

is an open mapping and

$$\mathrm{Id} : (S, \tau_2) \to (S, \tau_1)$$

is continuous.

Proposition 3.50 Let τ_1, τ_2 be topologies on a set S with $\tau_1 \subset \tau_2$. If (S, τ_1) is Hausdorff then so is (S, τ_2).

Proof. This is obvious by the inclusion of the topologies. ■

Proposition 3.51 Let τ_1, τ_2 be topologies on a set S with $\tau_1 \subset \tau_2$. If (S, τ_1) is Hausdorff and (S, τ_2) is compact then $\tau_1 = \tau_2$.

Comment 3.12 This means that one cannot weaken a compact Hausdorff topology without losing the Hausdorff property. This also means that one cannot strengthen a compact Hausdorff space without losing compactness.

Proof. Let F be a τ_2-closed set. Since (S, τ_2) is compact then F is τ_2-compact. Since $\tau_1 \subset \tau_2$ it follows that F is τ_1-compact (any τ_1-open cover of F is also a τ_2-open cover of F and has a finite subcover). Since τ_1 is Hausdorff and F is τ_1-compact then it is also τ_1-closed, which completes the proof (we showed that every τ_2-closed set is a τ_1-closed set). ■

Let S be a set and let $\mathscr{F}$ be a family of mappings from S into topological spaces:

$$\mathscr{F} = \{ f_\alpha : S \to Y_\alpha \mid \alpha \in I \}.$$

Let τ be the topology generated by the **subbase**

$$\{ f_\alpha^{-1}(V) \mid \alpha \in I, V \in \tau_{Y_\alpha} \}.$$

Then τ is the weakest topology on S for which all the f_α are continuous maps (it is the intersection of all topologies having this property). It is called the **weak topology induced by** $\mathscr{F}$, or the $\mathscr{F}$-topology of S.

Proposition 3.52 Let $\mathscr{F}$ be a family of mappings $S \to Y_\alpha$ where S is a set and each Y_α is a Hausdorff topological space. If $\mathscr{F}$ separates points on S then the $\mathscr{F}$-topology on S is Hausdorff.

Comment 3.13 Here separates between point means that $x \neq y$ implies that $f(x-y) \neq f(0)$. In the linear case it reduces to $f(x) \neq f(y)$.

Proof. Let $x \neq y \in S$. Since $\mathscr{F}$ separates points, there exists an $f \in \mathscr{F}$ such that

$$f(x-y) \neq f(0).$$

Because Y is Hausdorff there exists $U \in \mathscr{N}_{f(x-y)}$ and $W \in \mathscr{N}_{f(0)}$ that are disjoint. By definition $f^{-1}(U)$ and $f^{-1}(W)$, which are neighborhoods of $x-y$ and 0, are $\mathscr{F}$-open. The sets

$$x + f^{-1}(U) \qquad \text{and} \qquad y + f^{-1}(W)$$

are disjoint and open. ∎

Proposition 3.53 Let (S,τ) be a compact topological space. If there is a sequence $\{f_n \mid n \in \mathbb{N}\}$ of continuous real-valued functions that separates points in S then S is metrizable.

Proof. Since (S,τ) is compact and the f_n are continuous then they are bounded. Thus, we can normalize them such that $|f_n| \le 1$. Define:

$$d(x,y) = \sum_{n=1}^{\infty} \frac{|f_n(x) - f_n(y)|}{2^n}.$$

This series converges. In fact, it converges uniformly on $S \times S$ hence the limit is continuous. Because the f_n separate points $d(x,y) = 0$ iff $x = 0$. d is also symmetric and satisfies the triangle inequality.

Thus d is a metric and we denote by τ_d the topology induced by this metric. We need to show that $\tau_d = \tau$. Consider the metric balls:

$$\mathfrak{B}(x,r) = \{y \in S \mid d(x,y) < r\}.$$

Since d is τ-continuous on $S \times S$ these balls are τ-open and

$$\tau_d \subset \tau.$$

By Proposition 3.51, since τ is compact and τ_d is Hausdorff (like any metric space) then $\tau = \tau_d$. ∎

Lemma 3.54 Let $f_1,\dots,f_n$ and f be linear functionals on a vector space $\mathscr{V}$ (no topology). Let

$$N = \ker f_1 \cap \cdots \cap \ker f_n.$$

Then the following are equivalent:

① There are scalars such that

$$f = \sum_{k=1}^{n} \alpha_k f_k,$$

i.e.,

$$f \in \operatorname{Span}\{f_k \mid k = 1,\dots,n\}.$$

② There exists a constant $C > 0$ such that for all $x \in \mathscr{V}$:

$$|f(x)| \le \max_k |f_k(x)|.$$

③ $N \subset \ker f$.

Proof. Suppose that ① holds. Then for all x

$$|f(x)| \le \sum_{k=1}^{n} |\alpha_k||f_k(x)| \le \left(n \max_k |\alpha_k|\right) \max_k |f_k(x)|.$$

Suppose that ② holds:

$$|f(x)| \le C \max_k |f_k(x)|,$$

then f vanishes on N.

Finally, suppose that ③ holds. Define $T : \mathscr{V} \to \mathscr{F}^n$,

$$T(x) = (f_1(x),\dots,f_n(x)).$$

If is a linear map. If $T(x) = T(y)$ then $y - x \in N$ and $f(x) = f(y)$. Define a map

$$\sigma : T(\mathscr{V}) \to \mathscr{F}$$

by

$$\sigma(f_1(x),\dots,f_n(x)) = f(x).$$

σ can be extended as a linear functional to all of $\mathscr{F}^n$, i.e., there are scalars $\alpha_1,\dots,\alpha_n$ such that

$$\sigma(u_1,\dots,u_n) = \sum_{k=1}^{n} \alpha_k u_k,$$

and in particular

$$f(x) = \sigma(f_1(x),\dots,f_n(x)) = \sum_{k=1}^{n} \alpha_k f_k(x).$$

■

Theorem 3.55 Let $\mathscr{V}$ be a vector space (no topology) and let $\mathscr{V}'$ be a separating vector space of linear functionals on $\mathscr{V}$. Denote by τ' the $\mathscr{V}'$-topology on $\mathscr{V}$. Then $(\mathscr{V},\tau')$ is a locally convex topological vector space whose dual space is $\mathscr{V}'$.

Comment 3.14 Local convexity is important because of the Hahn-Banach theorem. An extension of the separation theorem states that If $\mathscr{X}$ is a locally convex topological vector space, A is compact, and B closed, then there exists a continuous linear map $f : \mathscr{X} \to \mathscr{F}$ and $s,t \in \mathbb{R}$ such that

$$\operatorname{Re} f(a) < t < s < \operatorname{Re} f(b) \qquad \text{for all } a \in A \text{ and } b \in B.$$

Proof. $\mathscr{V}'$ is a point-separating family of maps $\mathscr{V} \to \mathscr{F}$. Since $\mathscr{F}$ is a Hausdorff space, it follows from Proposition 3.52 that the $\mathscr{V}'$-topology on $\mathscr{V}$ is a Hausdorff topology. The topology τ' is translation invariant because the open sets in $(\mathscr{V},\tau')$ are generated by the base

$$\{f^{-1}(A) \mid f \in \mathscr{V}',\ A \text{ open in } \mathscr{F}\}.$$

and f is linear.

The topology τ' is generated by the local subbase,

$$V(f,r) = \{x \in \mathscr{V} \mid |f(x)| < r\}.$$

These sets are balanced and convex (by linearity) hence every finite intersection of them (i.e. every element in the local base) is balanced and convex. If follows that $(\mathscr{V},\tau')$ is a locally convex topological space.

We next show that $(\mathscr{V},\tau')$ is a topological vector space. Note that

$$\frac{1}{2}V(f,r) + \frac{1}{2}V(f,r) = \{\tfrac{1}{2}x + \tfrac{1}{2}y \mid |f(x)| < r,\ |f(y)| < r\} \subset V(f,r).$$

It follows that every U in the base satisfies such an inclusion as well, hence addition is continuous. Scalar multiplication is continuous by a similar argument.

It remains to show that $\mathscr{V}'$ is the dual space of $(\mathscr{V},\tau')$. Every $f \in \mathscr{V}'$ is τ'-continuous as for all $\varepsilon > 0$

$$|f(V(f,\varepsilon))| = |f(\{x \mid |f(x)| < \varepsilon\})| \subset [0,\varepsilon).$$

Conversely, let f be a τ'-continuous functional on $\mathscr{V}$; we need to show that it belongs to $\mathscr{V}'$. Take $\varepsilon = 1$, there exists a U in the local base such that

$$|f(U)| \subset [0,1).$$

In other words, there exists a U of the form

$$U = \{x \mid |f_1(x)| < r_1, \dots, |f_m(x)| < r_m\},$$

for which

$$\sup_{x \in U} |f(x)| < 1.$$

In particular,

$$\ker f \subset \ker f_1 \cap \cdots \cap \ker f_m.$$

It follows from that last lemma that $f \in \operatorname{Span}\{f_1, \dots, f_m\}$. ■

3.12.2 The weak topology of a topological vector space

Given a topology, we can determine whether a function is continuous. This argument can be reversed: given a space and functions on that space, we can define a topology with respect these functions are continuous. For example, we can take the discrete topology, which is not interesting (only sequence that are eventually constant converge). In the previous section we have laid the foundations to endow the space with the weakest topology that makes those function continuous. This is the context of weak topologies over topological vector spaces.

Weak and original topologies

Definition 3.25 Let $(\mathscr{X}, \tau)$ be a topological vector space whose dual $\mathscr{X}^*$ (the vector space of continuous linear functionals) separates points. The $\mathscr{X}^*$-topology on $\mathscr{X}$ is called the **weak topology** (it is the weakest topology with respect every τ-continuous linear functional is continuous).

We denote the weak topology by τ_w (the space itself is often denoted by $\mathscr{X}_w$).

Corollary 3.56 $\mathscr{X}_w$ is locally convex and $\mathscr{X}_w^* = \mathscr{X}^*$.

Proof. This is in fact what Theorem 3.55 states. ■

Another corollary is:

Corollary 3.57 $(\mathscr{X}_w)_w = \mathscr{X}_w$.

Note that since every $f \in \mathscr{X}^*$ is τ continuous and since τ_w is the weakest topology with this property, it follows that

$$\tau_w \subset \tau,$$

justifying the name of weak topology.

The next proposition shows that weak convergence is consistent with what we know:

Proposition 3.58 A sequence x_n in a topological vector space $(\mathscr{X}, \tau)$ weakly converges to zero, $x_n \rightharpoonup 0$, if and only if

$$\forall f \in \mathscr{X}^* \qquad f(x_n) \to 0.$$

Proof. Weak convergence means that for every τ_w-neighborhood V of zero, the sequence is eventually in V. Since every τ_w-neighborhood of zero contains a set of the form

$$\{x \mid |f_j(x)| < r_j,\ j = 1, \dots, n\},$$

if follows that $f(x_n) \to 0$ for all $f \in \mathscr{X}^*$ guarantees weak convergence.

Conversely, if $x_n \rightharpoonup 0$, then the sequence is eventually in every τ_w-neighborhood of zero, and in particular, for all $f \in \mathscr{X}^*$ and $\varepsilon > 0$, the sequence is eventually in

$$V(f, \varepsilon) = \{x \mid |f(x)| < \varepsilon\},$$

hence $f(x_n) \to 0$. ∎

Corollary 3.59 — Strong convergence implies weak convergence. Every τ-convergent sequence is τ_w convergent.

Proof. If $x_n \to 0$ then $f(x_n) \to 0$ for all $f \in \mathscr{X}^*$. ∎

Weak and original boundedness

Proposition 3.60 Let $(\mathscr{X}, \tau)$ be a topological vector space. A set $E \subset \mathscr{X}$ is τ_w-bounded (weakly bounded) if and only if

$$\forall f \in \mathscr{X}^* \qquad f \text{ is a bounded functional on } E.$$

Proof. E is weakly bounded iff (by definition) for all τ_w-neighborhoods V of zero

$$E \subset tV$$

for sufficiently large t. This means that for every set of the form

$$\{x \in \mathscr{X} \mid |f_j(x)| < r_j,\ j = 1, \dots, n\},$$

and for sufficiently large t:

$$E \subset \{tx \in \mathscr{X} \mid |f_j(x)| < r_j,\ j = 1, \dots, n\} = \{y \in \mathscr{X} \mid |f_j(y)| < tr_j,\ j = 1, \dots, n\},$$

which means that all those f_j are bounded on E. ■

Proposition 3.61 If $(\mathscr{X}, \tau)$ is an infinite-dimensional topological vector space then every τ_w-neighborhood of zero contains an infinite-dimensional subspace; in particular $(\mathscr{X}, \tau_w)$ is not locally bounded.

Comment 3.15 Recall that a topological vector space is metrizable if and only if it has a countable local base. Local boundedness implies the existence of a countable local base. Not being locally bounded does not imply a lack of metrizability.

Proof. Consider elements of the base

$$V = \{x \in \mathscr{X} \mid |f_j(x)| < r_j,\ j = 1, \dots, n\}.$$

Set as before

$$N = \ker f_1 \cap \dots \cap \ker f_n.$$

The map

$$x \mapsto (f_1(x), \dots, f_n(x))$$

is a map $\mathscr{X} \to \mathscr{F}^n$ with null space N. Hence,

$$\dim \mathscr{X} \le n + \dim N,$$

i.e., $\dim N = \infty$. Since $N \subset V$ it means that every element in the base contains an infinite-dimensional subspace. ■

Weak and original closedness

We next come to the concept of closure. If a set is τ_w-closed then it is clearly τ-closed. Let E be a set in a topological vector space $(\mathscr{X}, \tau)$. Its τ-closure $\overline{E}$ is the intersection of all τ-closed sets that contain it, whereas its τ_w-closure $\overline{E}_w$ is the intersection of all τ_w-closed sets that contain it. Since there are more τ-closed sets than τ_w-closed sets,

$$\overline{E} \subset \overline{E}_w.$$

Theorem 3.62 Let E be a convex subset of a locally convex topological vector space $(\mathscr{X}, \tau)$. Then,

$$\overline{E} = \overline{E}_w.$$

Proof. The proof of this theorem relies on a version of the Hahn-Banach separation theorem: if A, B are convex disjoint non-empty sets in a locally convex topological vector space, A is compact and B is closed, then there exists an $f \in \mathscr{X}^*$ such that

$$\sup\{\operatorname{Re} f(a) \mid a \in A\} < \inf\{\operatorname{Re} f(b) \mid b \in A\}.$$

Let $x_0 \notin \overline{E}$. It follows from the Hahn-Banach separation theorem that there exists an $f \in \mathscr{X}^*$ and a $\gamma \in \mathbb{R}$ such that

$$\operatorname{Re} f(x_0) < \gamma < \operatorname{Re} f(\overline{E}).$$

The set

$$\{x \mid \operatorname{Re} f(x) < \gamma\}$$

is a weak neighborhood of x_0 that does not intersect E, i.e.,

$$x \notin \overline{E}_w,$$

namely $\overline{E}_w \subset \overline{E}$. ∎

Comment 3.16 This last theorem states that if E is a convex set in a locally convex topological vector space and there is a sequence $x_n \in E$ that weakly converges to x (which is not necessarily in E), then there is also a sequence $y_n \in E$ that originally-converges to x. We proved such a theorem for Hilbert spaces (using the Banach-Saks theorem).

Corollary 3.63 For convex subsets of locally convex topological vector spaces:

① τ-closed equals τ_w-closed.
② τ-dense equals τ_w-dense.

Proof. It is always true that τ_w-closed implies τ-closed. If E is a τ-closed convex subset of a locally convex set then $\overline{E}_w = \overline{E} = E$, i.e., it is also τ_w-closed. The second part is then obvious. ∎

3.12.3 The weak-* topology of the dual space

Let $(\mathscr{X},\tau)$ be a topological vector space. The dual space $\mathscr{X}^*$ does not come with an a priori topology. In Banach spaces, the dual space has a natural operator norm, and we proved that the dual space endowed with that norm is also a Banach space (in fact, we proved that the dual of a normed space is a Banach space). But for general topological vector spaces, we don't have as for now a topology.

Recall the natural inclusion from $\mathscr{X}$ to linear functionals on $\mathscr{X}^*$,

$$\iota : x \to F_x,$$

where for $f \in \mathscr{X}^*$:

$$F_x(f) = f(x).$$

The family of functionals $\{\iota(x) \mid x \in \mathscr{X}\}$ (which we can't call continuous because there is no topology on $\mathscr{X}^*$) separates points on $\mathscr{X}^*$ as if

$$\iota(x)(f) = \iota(x)(g)$$

for all $x \in \mathscr{X}$ then

$$\forall x \in \mathscr{X} \qquad f(x) = g(x),$$

i.e., $f = g$.

If follows from Theorem 3.55 that the $\iota(\mathscr{X})$-topology of $\mathscr{X}^*$ turns it into a locally convex topological vector space whose dual space is $\iota(\mathscr{X})$. The $\iota(\mathscr{X})$-topology of $\mathscr{X}^*$ is called the **weak-* topology**. Every linear functional on $\mathscr{X}^*$ that is weak-* continuous is of the form $\iota(x)$ for some $x \in \mathscr{X}$. The open sets in the weak star topology $(\mathscr{X},\tau_*)$ are generated by the subbase:

$$V(x,r) = \{f \in \mathscr{X}^* \mid |f(x)| < r\}.$$

Weak-* convergence of a sequence $(f_n) \subset \mathscr{X}^*$ to $f \in \mathscr{X}^*$ means that

$$\forall x \in \mathscr{X} \qquad \lim_{n\to\infty} f_n(x) = f(x).$$

■ **Example 3.2** Recall that $c_0^* = \ell_1$ and $\ell_1^* = \ell_\infty$. Weak convergence of a sequence $(x_n)_k \subset \ell_1$ to zero (with ℓ_1 viewed as a topological vector space) means that

$$\forall y_k \in \ell_\infty \qquad \lim_{n\to\infty} \sum_{k=1}^{\infty} (x_n)_k y_k = 0.$$

Weak-* convergence of a sequence $(x_n)_k \in \ell_1$ to zero (with ℓ_1 viewed as the dual space of the topological vector space c_0) means that

$$\forall y_k \in c_0 \qquad \lim_{n\to\infty} \sum_{k=1}^{\infty} (x_n)_k y_k = 0.$$

Clearly, weak convergence implies weak-* convergence (but not the opposite). In reflexive spaces the two notions of convergence are identical. ■

The following central theorem states a compactness property of the weak-* topology. It was proved in 1932 by Banach for separable spaces and in 1940 by Alaoglu in the general case. (Leonidas Alaoglu (1914–1981) was a Greek mathematician.)

Theorem 3.64 — Banach-Alaoglu. Let $(\mathscr{X},\tau)$ be a topological vector space. Let $\mathscr{N}_0 \ni V \subset \mathscr{X}$ and let

$$K = \{f \in \mathscr{X}^* \mid |f(x)| \le 1 \quad \text{for all } x \in V\}.$$

Then K is weak-*-compact. (The set of functionals K is call the **polar** (הקבוצה הקוטבית) of the set of vectors V.)

Proof. Since V is absorbing (like any neighborhood of zero), every $x \in \mathscr{X}$ has a $\gamma(x) > 0$ such that

$$x \in \gamma(x)V.$$

Hence, for $x \in \mathscr{X}$ and $f \in K$:

$$|f(x)| = \gamma(x)|f(x/\gamma(x))| \le \gamma(x),$$

where we used the definition of K.

Let

$$D_x = \{\alpha \in \mathscr{F} \mid |\alpha| \le \gamma(x)\},$$

and let

$$P = \prod_{x \in \mathscr{X}} D_x$$

with the product topology τ_P (the weakest topology with respect to which the projections $\pi_x : P \to D_x$ are continuous). Each D_x is compact, and by Tychonoff's theorem (which relies on the axiom of choice), P is compact. Every element in P can be identified with a function $f : \mathscr{X} \to \mathscr{F}$ (not necessarily linear) satisfying

$$|f(x)| \le \gamma(x).$$

If follows that every $f \in K$ belongs to P, namely,

$$K \subset \mathscr{X}^* \cap P.$$

The set K can therefore be assigned two topologies: the weak-* topology τ_* induced from $\mathscr{X}^*$ and the product topology τ_P induced from P.

Lemma 3.65 τ_* and τ_P coincide on K.

Proof. Fix $f_0 \in K$. Take $x_1, \ldots, x_n \in \mathscr{X}$. Take $\delta > 0$. Set

$$W_1 = \{f \in \mathscr{X}^* \mid |f(x_j) - f_0(x_j)| < \delta,\ j = 1, \ldots, n\},$$

and

$$W_2 = \{f \in P \mid |f(x_j) - f_0(x_j)| < \delta,\ j = 1, \ldots, n\}.$$

Sets of the form W_1 and W_2 are local bases for τ_* and τ_P. Since

$$W_1 \cap K = W_2 \cap K$$

the two topologies coincide on K. ■

Lemma 3.66 K is a τ_P-closed subset of P.

Proof. Let f_0 belong to the τ_P closure of K. We need to show that it belongs to K. Choose $x, y \in \mathscr{X}$, scalars α, β and $\varepsilon > 0$. Consider the τ_P neighborhood of f_0,

$$\{f \in P \mid |f(x) - f_0(x)| < \varepsilon,\ |f(y) - f_0(y)| < \varepsilon,\ |f(\alpha x + \beta y) - f_0(\alpha x + \beta y)| < \varepsilon\}.$$

By definition of the closure there is an element $f \in K$ that belongs to this set, i.e., it is a linear functional satisfying

$$|f(x) - f_0(x)| < \varepsilon \quad |f(y) - f_0(y)| < \varepsilon \quad \text{and} \quad |\alpha f(x) + \beta f(y) - f_0(\alpha x + \beta y)| < \varepsilon.$$

It is easy to see that it follows that

$$|f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y)| < (1 + |\alpha| + |\beta|)\varepsilon,$$

and since this holds for all ε, it follows that f_0 is linear.

Let $x \in V$ and consider the τ_P-neighborhood of f_0:

$$\{f \in P \mid |f(x) - f_0(x)| < \varepsilon\}.$$

This neighborhood intersects K, hence there exists an $f \in K$ such that

$$|f(x)| \le 1 \qquad \text{and} \qquad |f(x) - f_0(x)| < \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we conclude that $|f_0(x)| \le 1$ for all $x \in V$, i.e., $f_0 \in K$. ■

Back to the main theorem: Since P is compact and K is τ_P-closed, it is τ_P compact. Since both topologies coincide on K, then K is τ_*-compact. ■

The Banach-Alaoglu theorem holds in a very general setting. The following theorem holds in the case where the space is separable:

Theorem 3.67 Let $\mathscr{X}$ be a separable topological vector space. Let $K \subset \mathscr{X}^*$ be weak-* compact. Then K endowed with the weak-* topology is metrizable.

Comment 3.17 The claim is not that $\mathscr{X}^*$ endowed with the weak-* topology is metrizable. For example, this is not true in infinite-dimensional Banach spaces.

Proof. Take

$$\{x_n \mid n \in \mathbb{N}\}$$

a dense set in $\mathscr{X}$ and $F_n = \iota(x_n)$. By definition of the weak-* topology on $\mathscr{X}^*$, the functional F_n are weak-* continuous. Also, if for every n,

$$F_n(f) = F_n(g),$$

i.e.,

$$f(x_n) = g(x_n),$$

then $f = g$ (continuous functionals that coincide on a dense set).

Thus, $\{F_n \mid n \in \mathbb{N}\}$ is a countable family of continuous functionals that separate points in $\mathscr{X}^*$. It follows by Proposition 3.53 that K is metrizable. ■

Corollary 3.68 Let $V \in \mathscr{N}_0$ in a separable topological vector space $\mathscr{X}$. Let $f_n \in \mathscr{X}^*$ satisfy

$$|f_n(x)| \le 1 \qquad \forall n \in \mathbb{N}, x \in V.$$

Then there exists an $f \in \mathscr{X}^*$ and a subsequence f_{n_k} such that

$$\lim_{k\to\infty} f_{n_k}(x) = f(x)$$

for all $x \in \mathscr{X}$.

Proof. Compactness and sequential compactness are equivalent in metric spaces. ■

3.13 The Krein-Milman theorem

The Krein–Milman theorem is a proposition about convex sets in topological vector spaces. A particular case of this theorem, which can be easily visualized, states that given a convex polygon, one only needs the corners of the polygon to recover the polygon shape (the polygon is the convex hull of its vertices).

Definition 3.26 Let $\mathscr{V}$ be a vector space and $E \subset \mathscr{V}$. The **convex hull** (קמור) of E, denoted $\operatorname{Conv}(E)$, is the intersection of all the convex sets that contains E (equivalently, it is the set of convex combinations of elements in E).

Definition 3.27 Let $\mathscr{V}$ be a vector space and $E \subset \mathscr{V}$. The **closed convex hull** of E, denoted $\overline{\operatorname{Conv}}(E)$ is the closure of the convex hull of E.

Definition 3.28 Let $\mathscr{X}$ be a topological vector space. A set $E \subset \mathscr{X}$ is called **totally bounded** (חסום כליל) if to every $V \in \mathscr{N}_0$ corresponds a finite set F, such that

$$E \subset F + V.$$

(Compare with the parallel notion in a metric space.)

Theorem 3.69

① Let $A_1, \dots, A_n$ be compact convex sets in a topological vector space $\mathscr{X}$. Then

$$\operatorname{Conv}(A_1 \cup \dots \cup A_n)$$

is compact.

② Let $\mathscr{X}$ be a locally convex topological vector space. If $E \subset \mathscr{X}$ is totally bounded then $\operatorname{Conv}(E)$ is totally bounded as well.

③ If $\mathscr{X}$ is a Frechet space (i.e., locally convex, metrizable and complete) and $K \subset \mathscr{X}$ is compact then $\overline{\operatorname{Conv}}(K)$ is compact.

④ If $K \subset \mathbb{R}^n$ is compact then $\operatorname{Conv}(K)$ is compact.

Proof.

① Let $S \subset \mathbb{R}^n$ be the simplex

$$S = \{(s_1, \dots, s_n) \mid s_j \geq 0, \quad s_1 + \dots + s_n = 1\}.$$

Set $A = A_1 \times \dots \times A_n$, and define the function $f : S \times A \to \mathscr{X}$:

$$f(s,a) = \sum_{k=1}^n s_k a_k.$$

Consider the set $K = f(S,A)$. It is the continuous image of compact sets and it is therefore compact. Moreover,

$$K \supset \operatorname{Conv}(A_1 \cup \dots \cup A_n).$$

It is easy to show that K is convex, and since it includes all the A_k's it must in fact be equal to $\operatorname{Conv}(A_1 \cup \cdots \cup A_n)$.

② Let $U \in \mathscr{N}_0$. Because $\mathscr{X}$ is locally convex there exists a convex neighborhood V such that

$$V + V \subset U.$$

Since E is totally bounded there exists a finite set F such that

$$E \subset F + V \subset \operatorname{Conv}(F) + V.$$

Since the right hand side is convex

$$\operatorname{Conv}(E) \subset \operatorname{Conv}(F) + V.$$

By the first item $\operatorname{Conv}(F)$ is compact, therefore there exists a finite set F' such that

$$\operatorname{Conv}(F) = F' + V,$$

i.e.,

$$\operatorname{Conv}(E) \subset F' + V + V \subset F' + U,$$

which proves that $\operatorname{Conv}(E)$ is totally bounded.

③ In every metric space the closure of a totally bounded set is totally bounded, and if the space is complete it is compact. Since K is compact, then it is totally bounded. By the previous item $\operatorname{Conv}(K)$ is totally bounded and hence its closure is compact.

④ Let $S \subset \mathbb{R}^{n+1}$ be the convex simplex. One can show that $\operatorname{Conv}(K)$ is the image of the continuous map $S \times K^{n+1}$:

$$(s, x_1, \ldots, x_n) \mapsto \sum_{k=1}^{n+1} s_k x_k,$$

whose domain is compact.

■

Definition 3.29 Let $\mathscr{V}$ be a vector space and $K \subset \mathscr{V}$. A non-empty set $S \subset K$ is called an **extreme set** (קבוצת קיצון) for K if no point of S is an internal point of a segment whose endpoints are in K, except if both points are in S. I.e.

$$x, y \in K, \; t \in (0,1), \; tx + (1-t)y \in S \quad \text{implies} \quad x, y \in S.$$

Extreme points (נקודות קיצון) are extreme sets that consist of one point. The set of all extreme points of K is denoted by $\operatorname{Ext}(K)$.

Comment 3.18 Note that extreme sets and extreme points are a pure vector space concept.

Comment 3.19 Every set is an extreme set of itself.

■ **Example 3.3** The extreme points of a convex polygon are its vertices. The extreme points of a circle are the entire circle. ■

Theorem 3.70 — Krein-Milman, 1940. Let $\mathscr{X}$ be a topological vector space in which $\mathscr{X}^*$ separates between points. If $K \subset \mathscr{X}$ is a nonempty compact convex set, then

$$K = \overline{\operatorname{Conv}}(\operatorname{Ext}(K)).$$

I..e, K is the closed convex hull of its extreme points.

Comment 3.20 David Milman (1912–1982) was a Soviet and then an Israeli Mathematician. He is the father of Vitali Milman, a mathematician at Tel-Aviv University, and grandfather of Emanuel Milman, a mathematician at the Technion.

Comment 3.21 Locally convex topological vector spaces have the property that $\mathscr{X}^*$ separates points.

Proof. Let

$$\mathscr{P} = \{\text{compact extreme subsets of } K\}.$$

This set is not empty because $K \in \mathscr{P}$.

$\mathscr{P}$ is closed under non-empty intersections: For every $\mathscr{P}' \subset \mathscr{P}$ and non-empty

$$S = \bigcap_{A \in \mathscr{P}'} A$$

we have $S \in \mathscr{P}$. This is because

$$tx + (1-t)y \in S,$$

where $x,y \in K$ and $t \in (0,1)$ implies that $x,y \in A$ for every $A \in \mathscr{P}'$, i.e., $x,y \in S$. I.e., every non-empty intersection of compact extreme sets of K is a compact extreme set of K.

Let $S \in \mathscr{P}$ and $f \in \mathscr{X}^*$. Define

$$S_f = \{x \in S \mid \operatorname{Re} f(x) = \mu\},$$

where

$$\mu = \max_{x \in S} \operatorname{Re} f(x).$$

Since S is compact, S_f is non-empty; in fact, it is compact. Suppose

$$z = tx + (1-t)y \in S_f.$$

Because $S_f \subset S$, and the latter is an extreme set, $x, y \in S$. By linearity of f:

$$\mu = \operatorname{Re} f(z) = t \operatorname{Re} f(x) + (1-t) \operatorname{Re} f(y) \le \mu.$$

It follows that $\operatorname{Re} f(x) = \operatorname{Re} f(y) = \mu$, i.e., $x, y \in S_f$, which implies that $S_f \in \mathscr{P}$.

Choose again $S \in \mathscr{P}$. Let

$$\mathscr{P}_S = \{A \in \mathscr{P} \mid A \subseteq S\}.$$

$\mathscr{P}_S$ is non-empty (it contains S) and it can be partially ordered by reverse set inclusion. Take any maximal chain $\Omega \subset \mathscr{P}_S$, and let

$$M = \bigcap_{A \in \Omega} A.$$

This intersection has the finite intersection property hence its it is not empty. We know that $M \in \mathscr{P}_S$. By the maximality of Ω, there is no proper subspace of M that belongs to $\mathscr{P}_S$.

Let $f \in \mathscr{X}^*$. Since $M_f \subset M$ and M_f is non-empty we conclude that $M = M_f$, i,e, $\operatorname{Re} f$ is constant on M. Multiply by $\imath$ and we conclude that f is constant on M. Since $\mathscr{X}^*$ separates point, it follows that M is a singleton, i.e., an extreme point. Thus, for every compact extreme set S of K,

$$S \cap \operatorname{Ext}(K) \ne \varnothing.$$

Since K is convex

$$\operatorname{Conv}(\operatorname{Ext}(K)) \subset K.$$

Since it is also compact,

$$\overline{\operatorname{Conv}}(\operatorname{Ext}(K)) \subset K.$$

We need to show that this is in fact an equality.

Suppose, by contradiction that

$$x \in K \qquad \text{and} \qquad x \notin \overline{\operatorname{Conv}}(\operatorname{Ext}(K)).$$

Consider the disjoint, compact and convex sets:

$$\{x\} \qquad \text{and} \qquad \overline{\operatorname{Conv}}(\operatorname{Ext}(K)).$$

By a version of the Hahn-Banach theorem there exists an $f \in \mathscr{X}^*$ such that

$$\sup_{y \in \overline{\operatorname{Conv}}(\operatorname{Ext}(K))} \operatorname{Re} f(y) < \operatorname{Re} f(x).$$

Consider $K_f \in \mathscr{P}$. We have that

$$\overline{\operatorname{Conv}}(\operatorname{Ext}(K)) \cap K_f = \varnothing,$$

which contradicts the fact that K_f must contain an extreme point. ∎