PAC Learning and Online Learning

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PAC Learning

Domain $\mathcal{X}$, binary labels $\mathcal{Y} = \{-1, +1\}$,
hypothesis class $\mathcal{H} = \{h : (h : \mathcal{X} \to \mathcal{Y})\}$

(Fixed unknown) distribution $\mathcal{D}$ over domain $\mathcal{X}$

Labeled training data $(x_1, y_1), \ldots, (x_m, y_m) \in \mathcal{X} \times \mathcal{Y}$

The training set **distributed** according to $\mathcal{D}$: $S = (x_1, \ldots, x_m) \sim \mathcal{D}^m$

Realizability assumption: $\exists f \in \mathcal{H}$ that **correctly** determines the **labels** of all $x \in \mathcal{X}$, i.e., $\forall x_i \in \mathcal{X}$, $y_i = f(x_i)$.

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Loss of hypothesis $h \in \mathcal{H}$: $L_{\mathcal{D},f}(h) = \mathbb{P}\mathrm{r}_{x \sim \mathcal{D}}[h(x) \neq f(x)]$

Class $\mathcal{H}$ is **PAC learnable** if for all ε, δ, there is # samples = $m_{\mathcal{H}}(\varepsilon, \delta)$ and algorithm A so that for any $m \geq m_{\mathcal{H}}(\varepsilon, \delta)$, $\mathcal{D}$ and f,

$$\mathbb{P}\mathrm{r}_{S \sim \mathcal{D}^m}\left[L_{\mathcal{D},f}(A(S)) \leq \varepsilon\right] \geq 1 - \delta$$

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VC dimension :

- $\mathcal{H}$ **shatters** $C \subseteq \mathcal{X}$ if each of the $2^{|C|}$ possible labelings of C can be produced by some $h \in \mathcal{H}$.
- VC dimension of $\mathcal{H} = \sup\{|C| : \mathcal{H} \text{ shatters } C\}$

Agnostic PAC Learning

Domain $\mathcal{X}$, labels $\mathcal{Y}$, hypothesis class $\mathcal{H} = \{h : (h : \mathcal{X} \to \mathcal{Y})\}$

(Fixed unknown) distribution $\mathcal{D}$ over $\mathcal{X} \times \mathcal{Y}$

Training set $S = \{(x_1, y_1), \ldots, (x_m, y_m)\} \sim \mathcal{D}^m$

Loss of hypothesis $h \in \mathcal{H}$: $L_{\mathcal{D}}(h) = \mathbb{P}\mathrm{r}_{(x,y)\sim\mathcal{D}}[h(x) \neq y]$

Class $\mathcal{H}$ is **agnostically PAC learnable** if for all ε, δ, there is #samples = $m_{\mathcal{H}}(\varepsilon, \delta)$ and algorithm A so that for any $m \geq m_{\mathcal{H}}(\varepsilon, \delta)$ and $\mathcal{D}$,

$$\mathbb{P}\mathrm{r}_{S\sim\mathcal{D}^m}\left[L_{\mathcal{D}}(A(S)) \leq \varepsilon + \min_{f\in\mathcal{H}} L_{\mathcal{D}}(f)\right] \geq 1 - \delta$$

Empirical Risk Minimization (ERM): $\arg\min_{h\in\mathcal{H}} L_S(h)$

Uniform convergence: ERM on $\frac{\varepsilon}{2}$-representative training sets

For finite hypothesis class $\mathcal{H}$, $\lceil \frac{2\ln(2|\mathcal{H}|/\delta)}{\varepsilon^2} \rceil$ samples suffice for $\frac{\varepsilon}{2}$-representative training set.

Online Learning: Setting

Two actions: H and L (binary classification).

On each day $t = 1, \ldots, T$:

1. Learner **picks action** $i_t \in \{H, L\}$
2. Adversary **picks loss** vector $\boldsymbol{\ell}_t = (\ell_t^H, \ell_t^L) \in [0,1]^2$
3. Learner learns $\boldsymbol{\ell}_t$ and **incurs loss** $\ell_t^{i_t}$

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Goal is to minimize **regret** (loss wrt. **best fixed** action in hindsight):

$$\text{Regret}(T) = \sup_{\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_T} \left(\sum_{t=1}^{T} \ell_t^{i_t} - \min_{i \in \{H,T\}} \sum_{t=1}^{T} \ell_t^{i} \right)$$

(Online learning) algorithm is **no-regret** if $\text{Regret}(T)/T \to 0$ at $T \to \infty$

Online Learning: Follow the Leader

Follow the Leader (FTL):

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1. Deterministic action choice, given the past (randomness always helps against the unknown).
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Proof : loss for action i_t (chosen by the algorithm) $= 1$, and loss for other action $= 0$.

Any deterministic algorithm incurs loss $= T$, while best action incures loss $\leq T/2$.

Online Learning: Randomization

Two actions: H and L (binary classification).

On each day $t = 1, \ldots, T$:

1. Learner picks action H with probability p_t (and L with probability $1 - p_t$.
2. Adversary picks loss vector $\boldsymbol{\ell}_t = (\ell_t^H, \ell_t^L) \in [0, 1]^2$
3. Learner learns $\boldsymbol{\ell}_t$ and incurs **expected loss**

$$f(p_t; \boldsymbol{\ell}_t) = p_t \ell_t^H + (1 - p_t)\ell_t^L$$

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Goal is to minimize **expected regret** :

$$\text{Exp-Regret}(T) = \sup_{\boldsymbol{\ell}_1,\ldots,\boldsymbol{\ell}_T} \left(\sum_{t=1}^{T} f(p_t; \boldsymbol{\ell}_t) - \min_{p\in[0,1]} \sum_{t=1}^{T} f(p; \boldsymbol{\ell}_t) \right)$$

Randomization potentially allows for improved **stability** .

Online Learning: (Randomized) Follow the Leader

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$$p_t = \arg\min_{p \in [0,1]} \sum_{\tau=1}^{t-1} f(p; \boldsymbol{\ell}_\tau) = \arg\min_{p \in [0,1]} F_{t-1}(p)$$

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For any loss sequence $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_T$, FTL has:

$$\text{Exp-Regret}_{FTL}(T) = \underbrace{\sum_{t=1}^{T} f(p_t; \boldsymbol{\ell}_t) - \min_{p\in[0,1]} \sum_{t=1}^{T} f(p; \boldsymbol{\ell}_t)}_{\text{expected regret}} \le \underbrace{\sum_{t=1}^{T} |p_t - p_{t+1}|}_{\text{stability}}$$

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For the analysis, we define **Be the Leader** (BTL):

$$p_t^* = \arg\min_{p \in [0,1]} \sum_{\tau=1}^{t} f(p; \boldsymbol{\ell}_\tau) = \arg\min_{p \in [0,1]} F_t(p)$$

Regret of FTL Against BTL

Lemma : For any loss sequence $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_T$,

$$\text{Regret}_{FTL}(T) \leq \text{Regret}_{BTL}(T) + \underbrace{\sum_{t=1}^{T} |p_t - p_{t+1}|}_{\text{stability}}$$

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$$\begin{aligned}
\sum_{t=1}^{T} f(p_t; \boldsymbol{\ell}_t) &= \sum_{t=1}^{T} f(p_t^*; \boldsymbol{\ell}_t) + \sum_{t=1}^{T} \Big(f(p_t; \boldsymbol{\ell}_t) - f(p_t^*; \boldsymbol{\ell}_t) \Big) \\
&= \sum_{t=1}^{T} f(p_t^*; \boldsymbol{\ell}_t) + \sum_{t=1}^{T} (p_t - p_t^*)(\ell_t^H - \ell_t^L) && \text{by dfn of } f(p_t; \boldsymbol{\ell}_t) \\
&\leq \sum_{t=1}^{T} f(p_t^*; \boldsymbol{\ell}_t) + \sum_{t=1}^{T} |p_t - p_t^*| && \text{losses } \boldsymbol{\ell}_t \in [0,1]^2 \\
&= \sum_{t=1}^{T} f(p_t^*; \boldsymbol{\ell}_t) + \sum_{t=1}^{T} |p_t - p_{t+1}| && \text{by dfn, } p_t^* = p_{t+1}
\end{aligned}$$

Regret of Be the Leader

Lemma: For any loss sequence $\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_T$, $\text{Regret}_{BTL}(T) \leq 0$

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By **induction** on t, we show that for any $t \geq 1$:

$$\underbrace{\sum_{\tau=1}^{t} f(p^*_\tau; \boldsymbol{\ell}_t)}_{\text{loss of BTL up to } t} \leq \underbrace{\min_{p \in [0,1]} F_t(p)}_{\text{loss of best fixed action up to } t} = \underbrace{F_t(p^*_t)}_{\text{by definition of } p^*_t}$$

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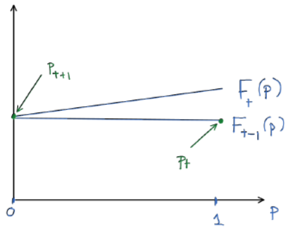
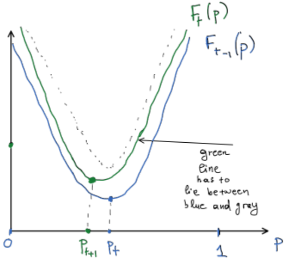
$$\begin{aligned}
\sum_{\tau=1}^{t+1} f(p^*_\tau; \boldsymbol{\ell}_\tau) &= f(p^*_{t+1}; \boldsymbol{\ell}_{t+1}) + \sum_{\tau=1}^{t} f(p^*_\tau; \boldsymbol{\ell}_\tau) \\
&\leq f(p^*_{t+1}; \boldsymbol{\ell}_{t+1}) + \min_{p \in [0,1]} F_t(p) && \text{induction hypth.} \\
&\leq f(p^*_{t+1}; \boldsymbol{\ell}_{t+1}) + F_t(p^*_{t+1}) && F_t(p^*_t) \leq F_t(p^*_{t+1}) \\
&= F_{t+1}(p^*_{t+1}) && \text{by dfn of } F_{t+1}(p)
\end{aligned}$$

Convexity and Stability



(a) Two linear functions that are close to each other can have very far minima.



(b) For convex functions, closeness of the functions implies closeness of their minima.

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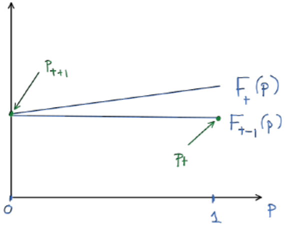

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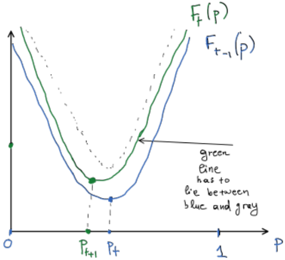
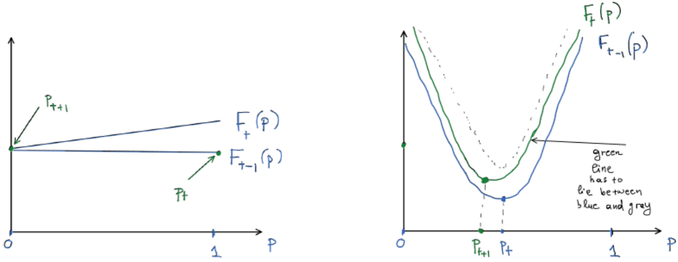


(b) For convex functions, closeness of the functions implies closeness of their minima.

$1/\eta$-**strongly convex** function $f : S \to \mathbb{R}$ wrt norm $\|\cdot\|$, if $\forall\, x, y \in S$:

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{1}{2\eta} \|x - y\|^2$$



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Functions $f, g : S \to \mathbb{R}$ be $1/\eta$-strongly convex wrt some norm $\|\cdot\|$ and $h(x) = g(x) - f(x)$ be L-Lipschitz wrt $\|\cdot\|$.

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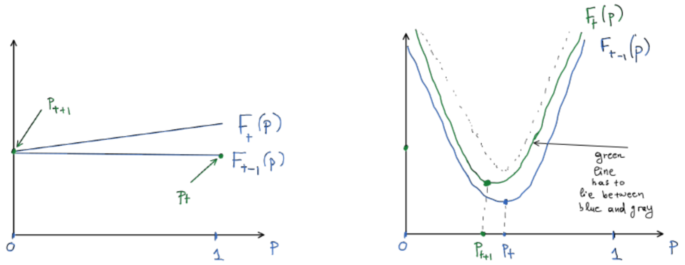

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Then, $\|x_f^* - x_g^*\| \leq \eta \cdot L$, with x_f^*, x_g^* minimizers of f, g.

Convexity and Stability

Functions $f, g : [0,1] \to \mathbb{R}$ be $1/\eta$-strongly convex and $h(x) = g(x) - f(x)$ be L-Lipschitz.

Then, $|p_f - p_g| \leq \eta \cdot L$, with p_f, p_g minimizers of f, g.

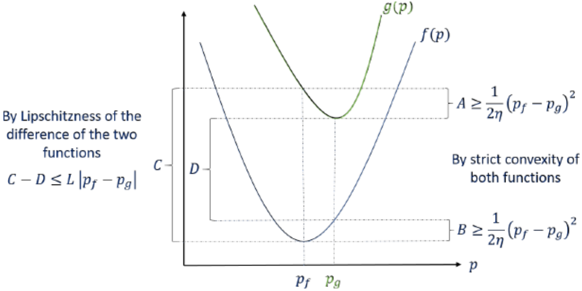

Figure 3: The proof of Lemma 3 follows immediately by noting that $C - D = A + B$ in the above figure, together with the fact that $C - D \leq L|p_f - p_g|$ by Lipschitzness of the difference of the two functions and $A + B \geq \frac{1}{\eta}(p_f - p_g)^2$ by the strict convexity of the two functions.

Convexity Through Regularization

If **cumulative loss** $F_t(\cdot)$ was $1/\eta$-strongly convex (for all t), stability could be bounded as:

$$\sum_{t=1}^{T} |p_t - p_{t+1}| \leq \eta \cdot T\,,$$

because $F_t(p) - F_{t-1}(p) = f(p; \boldsymbol{\ell}_t)$ is 1-Lipschitz (due to $\boldsymbol{\ell}_t \in [0,1]^2$).

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But our cumulative loss $F_t(\cdot)$ **is not** strongly convex!

Make it strongly convex through **regularization** !

$\tilde{F}_t(p) = F_t(p) + R(p)/\eta$, where $R(\cdot)$ any 1-strongly convex function:

- $R(p) = p^2/2$
- $R(p) = p\ln(p) + (1-p)\ln(1-p)$
- $R(p) = \ln(\frac{p}{1-p})$

Follow / Be the Regularized Leader

$F_t(p) = \sum_{\tau=1}^{t} f(p; \boldsymbol{\ell}_\tau)$ and $\tilde{F}_t(p) = \sum_{\tau=1}^{t} f(p; \boldsymbol{\ell}_\tau) + R(p)/\eta$

FTRL: $\tilde{p}_t = \arg\min_{p \in [0,1]} \tilde{F}_{t-1}(p)$

BTRL: $\tilde{p}_t^* = \arg\min_{p \in [0,1]} \tilde{F}_t(p)$

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Theorem :

$$\mathrm{Regret}_{FTRL}(T) \leq \eta \cdot T + \frac{2 \max_{p \in [0,1]} |R(p)|}{\eta}$$

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Let $R^* = \max_{p \in [0,1]} |R(p)|$.

Setting $\eta = \sqrt{2R^*/T}$, we get $\text{Regret}_{FTRL}(T) \leq 2\sqrt{2R^* T}$

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Lower bound on $\mathrm{Regret}_A(T)$ for any online (even randomized) optimization algorithm A?

Regret of FTRL Against BTRL

$$\begin{aligned}\text{Regret}_{FTRL}(T) &\leq \text{Regret}_{BTRL}(T) + \sum_{t=1}^{T} |\tilde{p}_t - \tilde{p}_{t+1}| \\ &\leq \text{Regret}_{BTRL}(T) + \eta \cdot T\end{aligned}$$

Regret of FTRL Against BTRL

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Proof : Second inequality from strong convexity, because $\tilde{p}_t, \tilde{p}_{t+1}$ are minimizers of $1/\eta$-strongly convex functions $\tilde{F}_{t-1}(p)$ and $\tilde{F}_t(p)$ with difference $f_t(p)$ which is 1-Lipschitz.

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$$\begin{aligned}\text{Regret}_{FTRL}(T) &\leq \text{Regret}_{BTRL}(T) + \sum_{t=1}^{T} |\tilde{p}_t - \tilde{p}_{t+1}| \\ &\leq \text{Regret}_{BTRL}(T) + \eta \cdot T\end{aligned}$$

Proof : Second inequality from strong convexity, because $\tilde{p}_t, \tilde{p}_{t+1}$ are minimizers of $1/\eta$-strongly convex functions $\tilde{F}_{t-1}(p)$ and $\tilde{F}_t(p)$ with difference $f_t(p)$ which is 1-Lipschitz.

$$\begin{aligned}\text{Regret}_{FTRL}(T) - \text{Regret}_{BTRL}(T) &= \sum_{t=1}^{T} (f(\tilde{p}_t; \boldsymbol{\ell}_t) - f(\tilde{p}_t^*; \boldsymbol{\ell}_t)) \\ &\leq \sum_{t=1}^{T} |\tilde{p}_t - \tilde{p}_t^*| \\ &= L \sum_{t=1}^{T} |\tilde{p}_t - \tilde{p}_{t+1}|\end{aligned}$$

Regret of Be the Regularized Leader

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Proof: Let $f_t(p) = f(p; \boldsymbol{\ell}_t)$ for brevity.

- Let $f_0(p) = R(p)/\eta$ and $\tilde{p}_0^* = \arg\min_{p\in[0,1]} R(p)/\eta$.
- Using induction on t, we show that for all $t \ge 1$,

$$\sum_{\tau=0}^{t} f_\tau(\tilde{p}_\tau^*) \le \tilde{F}_t(\tilde{p}_t^*) \quad \text{(including fake action } \tilde{p}_0^* \text{ at } \tau = 0)$$

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- Hence, by rearranging:

$$\sum_{t=1}^{T} f_t(\tilde{p}_t^*) - \min_{p\in[0,1]} \sum_{t=1}^{T} f_t(p) \leq \max_{p\in[0,1]} R(p)/\eta - \min_{p\in[0,1]} R(p)/\eta \leq 2\max_{p\in[0,1]} |R(p)|/\eta$$

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Setting $\eta = \sqrt{2R^*/T}$, we get $\text{Regret}_{FTRL}(T) \leq 2\sqrt{2R^* T}$

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Multiplicative weight updates :

- Negative entropy $E^-(p) = p \ln(p) + (1-p)\ln(1-p)$ is 1-strongly convex wrt L_1 norm.
- Using $E^-(p)$ as regularizer, results in the following update rule for expected loss $f(p_t; \boldsymbol{\ell}_t) = p_t \ell_t^H + (1-p_t)\ell_t^L$:

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- If $\boldsymbol{\ell}_t \in [0,1]^2$, setting $\eta = \sqrt{\ln(2)/T}$, yields regret $2\sqrt{T \ln(2)}$