Algorithmic Game Theory

Truthful Mechanisms for Welfare Maximization

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Designing welfare maximizing truthful auctions for single parameter environments

Single parameter auctions

- For the single-item case, we saw that the Vickrey auction is ideal
- We would like to achieve the same properties for any other type of auction
  - truthfulness and individual rationality [incentive guarantees]
  - welfare maximization [economic performance guarantees]
  - implementation in polynomial time [computational performance guarantees]
- Can we achieve all 3 properties for any single-parameter environment?

Knapsack auctions

- We will see an illustration for knapsack auctions
- k identical items for sale
- Each bidder i has a publicly known demand for w_i items
  - Inelastic demand
  - The mechanism should either give w_i items to the bidder or should not give him anything
- Each bidder i submits a bid b_i for his value per unit
- Real value per unit = v_i
- Assume the demands $(w_1, w_2, ..., w_n)$ are known to the mechanism
  - Say bidders have no incentive to lie about them
- Only private information to bidder i is v_i

Knapsack auctions

Alternative view of knapsack auctions

- The auctioneer has a resource of total capacity k (a knapsack)
- Each bidder requires size w_i , if he is served
- Each bidder has a value v_i , if he is served
- The auctioneer needs to select a subset of bidders to serve so as not to exceed the capacity k

Feasible allocations:

- $(x_1, x_2, \ldots, x_n)$ with $x_i \in \{0, 1\}$, and $\sum_i w_i x_i <= k$
- Just like the feasible solutions of a knapsack problem

Knapsack auctions

Example

- Resource = the half-time break in the Champions League final
- Capacity k = total length of the break
- Each bidder corresponds to a company who wants to be advertised during the break
- The size w_i is the duration of the ad of bidder i
- The auctioneer needs to select a subset of bidders as winners and present their ads without exceeding the time capacity k

Knapsack auctions

- Let **b** = $(b_1, b_2, \ldots, b_n)$ be the biding vector
- Need to decide the allocation and payment rule
- For the allocation rule:
  - Think of maximizing the social welfare
  - Then we have precisely the 0-1 Knapsack problem!

$$\max \; \sum_i b_i x_i$$

s.t.

$$\sum_i w_i x_i <= k$$

$$x_i \in \{0, 1\}, \text{ for } i = 1, \ldots, n$$

Knapsack auctions

Claim: The allocation rule that maximizes the social welfare is monotone

•Consider a winner and see what can happen if he increases his bid

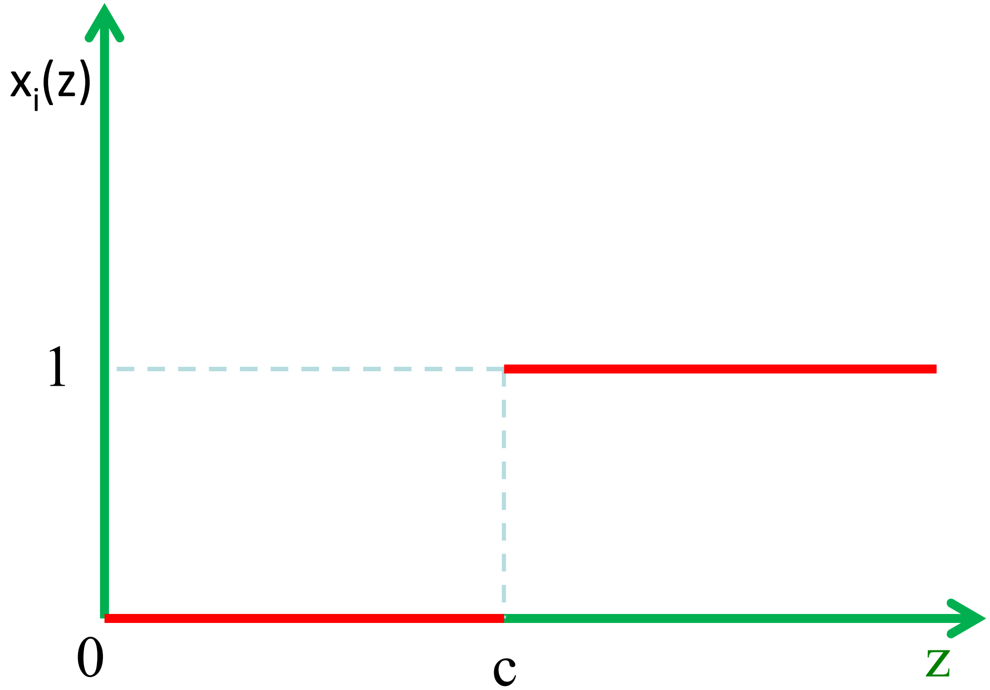
Hence, we can apply Myerson's lemma

How many jumps can we have for the allocation of a single player?

•At most one, a player can jump from being a loser ($x_i = 0$) to being a winner ($x_i = 1$)

Myerson's lemma and knapsack auctions

- The jump for a winner i happens at i's *critical bid*: the minimum he could bid and still be a winner, also known as *threshold bid*
- Generalization of the payment in Vickrey auction



Final mechanism:

- Solve the knapsack problem and find an optimal solution
- Give to each winner i, the requested number of items w_i
- Charge the winners their **critical bid**

Myerson's lemma and knapsack auctions

Does this mechanism achieve the desirable properties we wanted?

- truthfulness [YES]
- welfare maximization [YES]
- implementation in polynomial time [?]

•Knapsack is an NP-complete problem

•The properties can be enforced only for special cases where Knapsack is easy

- If highest bid or highest demand is polynomial in n (by dynamic programming)
- If weights form a super-increasing sequence

Algorithmic Mechanism Design

- The requirement for low complexity usually comes in conflict with the other criteria
- Goal of algorithmic mechanism design: explore the trade-offs between the 3 main properties (or any other properties that we may require in a given setting)
  - Truthfulness
  - welfare maximization
  - implementation in polynomial time
- Approach: relax one of the criteria and see if we can achieve the others
- For Knapsack and in general whenever welfare maximization is NP-complete: resort to approximation algorithms

Knapsack auctions

Goal for Knapsack:

- Find an approximation algorithm for the social welfare
- Prove that it is monotone

Recall:

Definition: An algorithm *A*, for a maximization problem, achieves an approximation factor of γ ($\gamma \leq 1$), if for every instance I of the problem, the solution returned by *A* satisfies:

$$SOL(I) \geq \gamma \, OPT(I)$$

Where OPT(I) is the value of the optimal solution for instance I

Knapsack auctions

- There are several heuristics and approximation algorithms for Knapsack, but not all of them are monotone
- A greedy ½-approximation:
  - For each bidder i, we care to evaluate the quantity b_i/w_i
  - Intuitively, we prefer bidders with small size/demand and large value
- Step 1: Sort and re-index the bidders so that

$$b_1/w_1 \geq b_2/w_2 \geq \ldots \geq b_n/w_n$$

- Step 2: Pick bidders in that order until the first time that adding someone exceeds the knapsack capacity
- Step 3: Return either the previous solution, or just the highest bidder if he achieves higher social welfare on his own

Knapsack auctions

- Why do we need the last step?
- Maybe there is a bidder with a very high value, but with a large demand as well
- The algorithm may not select this bidder in the first steps
- Step 3 ensures we do not miss out such highly-valued bidders

- Claim: This algorithm is monotone

- Theorem: Using Myerson's lemma, we can have a truthful polynomial time mechanism, that produces at least 50% of the optimal social welfare

Knapsack auctions

Going further

- Knapsack also admits an FPTAS (Fully Polynomial Time Approximation Scheme)
  - We can have a (1- ε)-approximation for any constant ε >0 [Ibarra, Kim '75]
  - But this is not a monotone algorithm
- [Briest, Krysta, Voecking '05]: A truthful FPTAS for Knapsack

- **Conclusion**: For a knapsack auction and any ε >0, we have a truthful mechanism that produces at least (1 – ε)-fraction of the optimal social welfare and runs in time polynomial in n and 1/ ε

General Approach

Suppose we have a single-parameter auction where the social welfare maximization problem is NP-hard

- Check if any of the known approximation algorithms for the problem is monotone (usually not)
- If not, then try to tweak it so as to make it monotone (sometimes feasible)
- Or design a new approximation algorithm that is monotone (hopefully without worsening the approximation guarantee)

Single-minded bidders

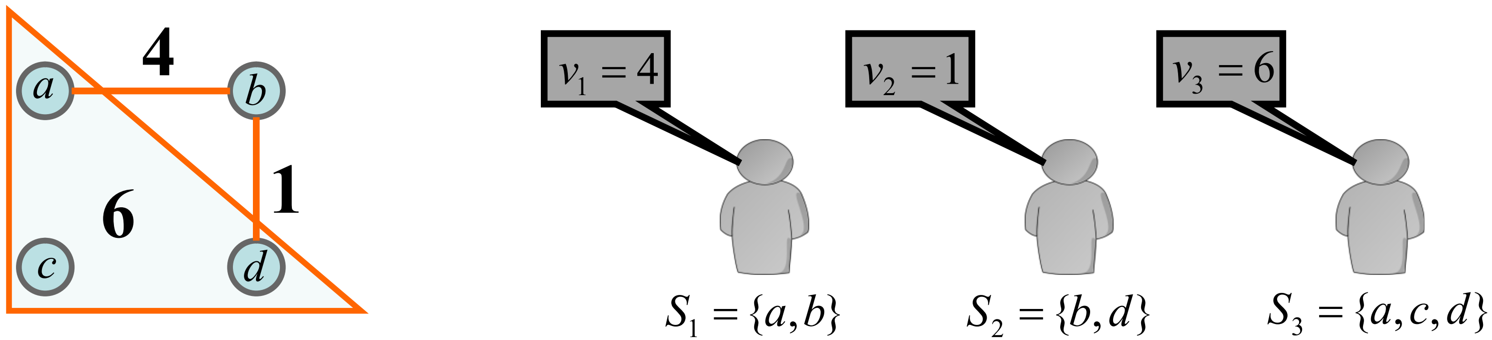
A single-paramerer auction with non-identical items

- The auctioneer has a set M of items for sale
- Each bidder i is interested in acquiring a specific subset of items, $S_i \subseteq M$ (known to the mechanism)
  - If the bidder does not obtain S_i (or a superset of it), his value is 0
- Each bidder submits a bid b_i for his value if he obtains the set

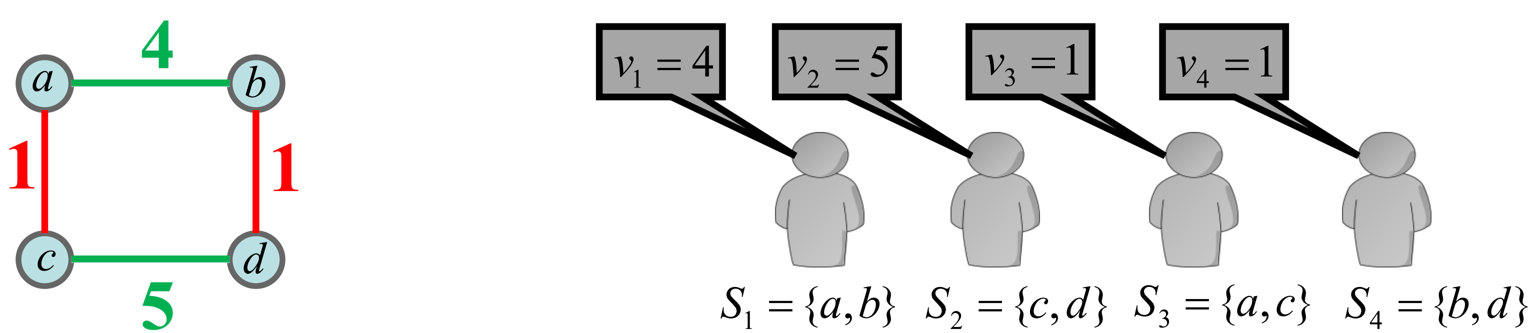
- Motivated by certain spectrum auctions
- Feasible allocations: the auctioneer needs to select winners who do not have overlapping sets

Single-minded bidders

Examples



- In the example above, the auctioneer can accept only 1 bidder as a winner
- In the example below, the auctioneer can accept up to 2 bidders as winners



Single-minded bidders

Social welfare maximization:

•Given the bids of the players, select a set of bidders with non-overlapping subsets, so as to maximize the sum of their bids

•It contains the SET PACKING problem, hence NP-hard

•Actually it gets even worse w.r.t. approximation

Theorem [Sandholm '99]: Under certain complexity theory assumptions, we cannot have an algorithm with approximation factor better than 1/sqrt(m)

Q: Can we have a 1/sqrt(m)-approximation?

Single-minded bidders

$s_i = |S_i|$

[Lehmann, O' Callaghan, Shoham '01]:

- Order the bidders in decreasing order of $b_i/\text{sqrt}(s_i)$
- Accept each bidder in this order unless overlapping with previously accepted bidders
- Payment i: largest bid b_j for set S_j with nonempty intersection with S_i .

- This algorithm achieves
  - 1/sqrt(m)-approximation, where m = |M|
  - 1/d-approximation, where $d = \max_i s_i$
  - Monotonicity and truthfulness.

Final conclusion: truthful polynomial time mechanism with the best possible approximation to the social welfare

Single-minded bidders

$s_i = |S_i|$

- Order the bidders in decreasing order of $b_i/\text{sqrt}(s_i)$
- Accept each bidder in this order unless overlapping with previously accepted bidders

- *A* algorithm's solution (set of indices accepted by Greedy)
- O optimal solution (set of indices accepted by OPT)

Wlog. assume that $O \cap A = \emptyset$.

Partition O into O_i, $i \in A$, s.t. $j \in O_i$ if $j \in O$ and $S_i \cap S_j \neq \emptyset$.

$$\begin{aligned}
\sum_{j \in O_i} v_j &\leq \frac{v_i}{\sqrt{s_i}} \sum_{j \in O_i} \sqrt{s_j} && \text{Greedy property} \\
&\leq \frac{v_i}{\sqrt{s_i}} \sqrt{\sum_{j \in O_i} s_j} \sqrt{|O_i|} && \text{Cauchy-Schwarz ineq.} \\
&\leq \frac{v_i}{\sqrt{s_i}} \sqrt{m} \sqrt{s_i} && |O_i| \leq s_i \text{ and } \textstyle\sum_{j \in O_i} s_j \leq m \\
&\leq v_i \sqrt{m}
\end{aligned}$$

Wrapping Up

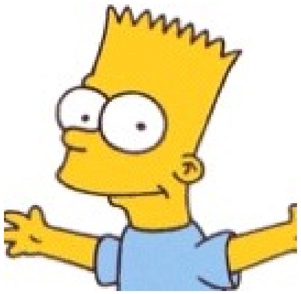
- Single parameter bidders: private information of bidder i is single value v_i , expressed by bid b_i
- Myerson's Lemma: truthful mechanism iff monotone allocation, payments are uniquely determined (and virtually always easy to compute).
  - 2^{nd} price / Vickrey auction is the **only** truthful single-item auction.
  - Optimal is always monotone: if allocation problem is easy, we also get computational efficiency.
  - If allocation problem is hard, we seek monotone poly-time approximation algorithms.
  - (1-1/k)-approximation in time $O(n^{k+1})$ and FPTAS for Knapsack (with demand known).
  - Single-minded bidders / set packing: Greedy wrt $b_i/sqrt(s_i)$ is monotone and O(sqrt(m))-approximation (best possible approximation in polynomial time).

Multi-dimensional Bidders / Combinatorial Auctions

The model

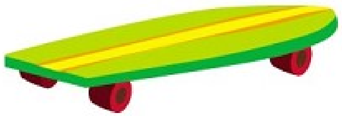
Set of players

N = {1, 2, …, n}



Set of indivisible goods

M = {1, 2, …, m}



Combinatorial Auctions

- Any auction with multiple items for sale
- The players may be allowed to express interest / bids on various combinations of goods
- In practice very active field within the last 10-15 years
  - Spectrum licenses
  - The FCC incentive auction:
    - https://www.fcc.gov/about-fcc/fcc-initiatives/incentive-auctions
  - Transportation routes
  - Logistics

Combinatorial auctions

- In practice, it seems economically more efficient and profitable to sell the items **together** than have a separate auction for each good
- Main challenges:
  - Algorithmic: How shall we design the allocation rule (especially if we have many overlaps in what the players want the most)?
  - Game-theoretic: Can we generalize Myerson's lemma to get truthful mechanisms?

Valuation functions

- So far we studied settings where a single parameter v_i determined all the information we needed for a player
- Most general scenario: consider that each player has a valuation function defined for every subset of the items
- $v_i : P(M) \rightarrow R$
  - where P(M) = powerset of M (all subsets of M)
  - For every $S \subseteq M$,
    - $v_i(S)$ = value of player i if he acquires set S

      $b_i(S)$ = maximum amount willing to pay for acquiring S
- We always assume **monotonicity** ("free-disposal"): for all $T \subseteq S$, $v_i(T) \leq v_i(S)$.

Examples of valuation functions

Additive valuation functions

- For every $S \subseteq M$, $v_i(S) = \sum_{j \in S} v_{ij}$
  - where v_{ij} = utility of acquiring item j
- Hence, the function can be completely determined by specifying the vector $(v_{i1}, v_{i2}, ..., v_{im})$
- m parameters for each bidder
- In such cases, the goods can be auctioned independently:
  - The value of an item is not affected by other items that a bidder may have already obtained

Examples of valuation functions

- In practice, the items may be interrelated with each other and additive valuations are not appropriate
- The value they add to a player may depend on the other items that the player has
- The items may exhibit
  - **Complementarity**: some items may be valuable only when they are sold together with other items (e.g. left and right shoe)
  - **Substitutability:** some items may be of similar type and should not be sold together to the same player (e.g. 2 cars with the same features)

Examples of valuation functions

Subadditive functions

- For any 2 disjoint subsets $S \subseteq M$, $T \subseteq M$,

$$v_i(S \cup T) \leq v_i(S) + v_i(T)$$

- In this case, we have substitutability among the goods
- They are also called complement-free functions (since we do not have complementarity)

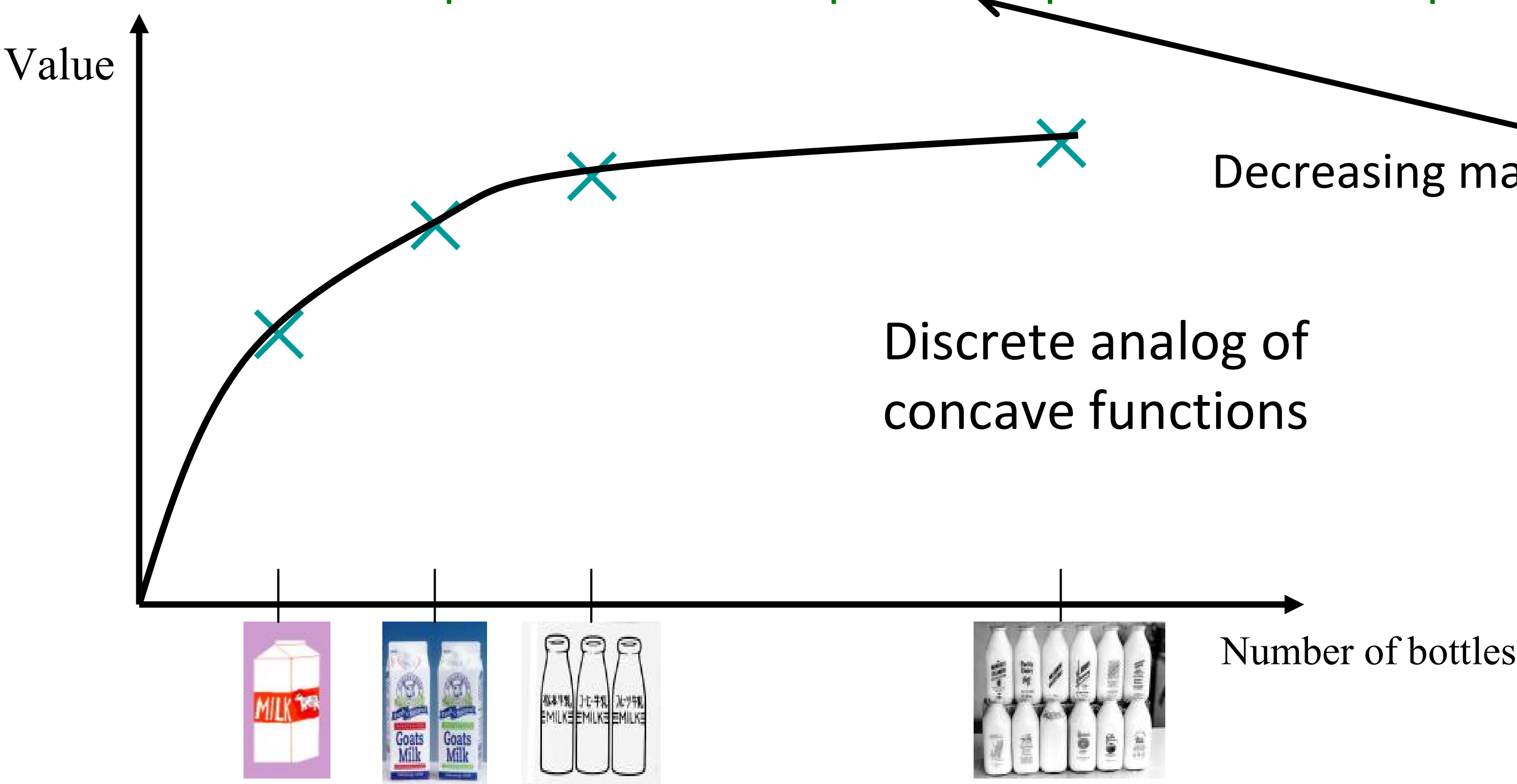
Examples of valuation functions

Submodular functions

For any 2 subsets S, T, with $S \subseteq T \subseteq M$, and for every $j \notin T$

$$v_i(T \cup \{j\}) - v_i(T) \leq v_i(S \cup \{j\}) - v_i(S)$$

Decreasing marginal values



Examples of valuation functions

- Submodular functions form a special class of subadditive valuations
- Hence, they also do not exhibit complementarity
- They play a key role in micro-economic theory
- Expressing the fact that utility gets “saturated” as we keep allocating substitutes to the same player

Examples of valuation functions

Symmetric submodular

- Special case of submodular functions, where all goods are identical
  - Hence, the final utility depends only on how many items the player receives
- Applicable for multi-unit auctions
  - E.g., auctions for government bonds fall under this framework
- For k identical items, such functions can be represented by a vector of k marginal values
  - $(m_i(1), m_i(2), \ldots, m_i(k))$ with $m_i(j) \geq m_i(j+1)$
  - Where $m_i(j)$ = additional utility to the player for obtaining the j-th unit, if the player already has j-1 units

Examples of valuation functions

Superadditive functions

•For any 2 disjoint subsets $S \subseteq M$, $T \subseteq M$,

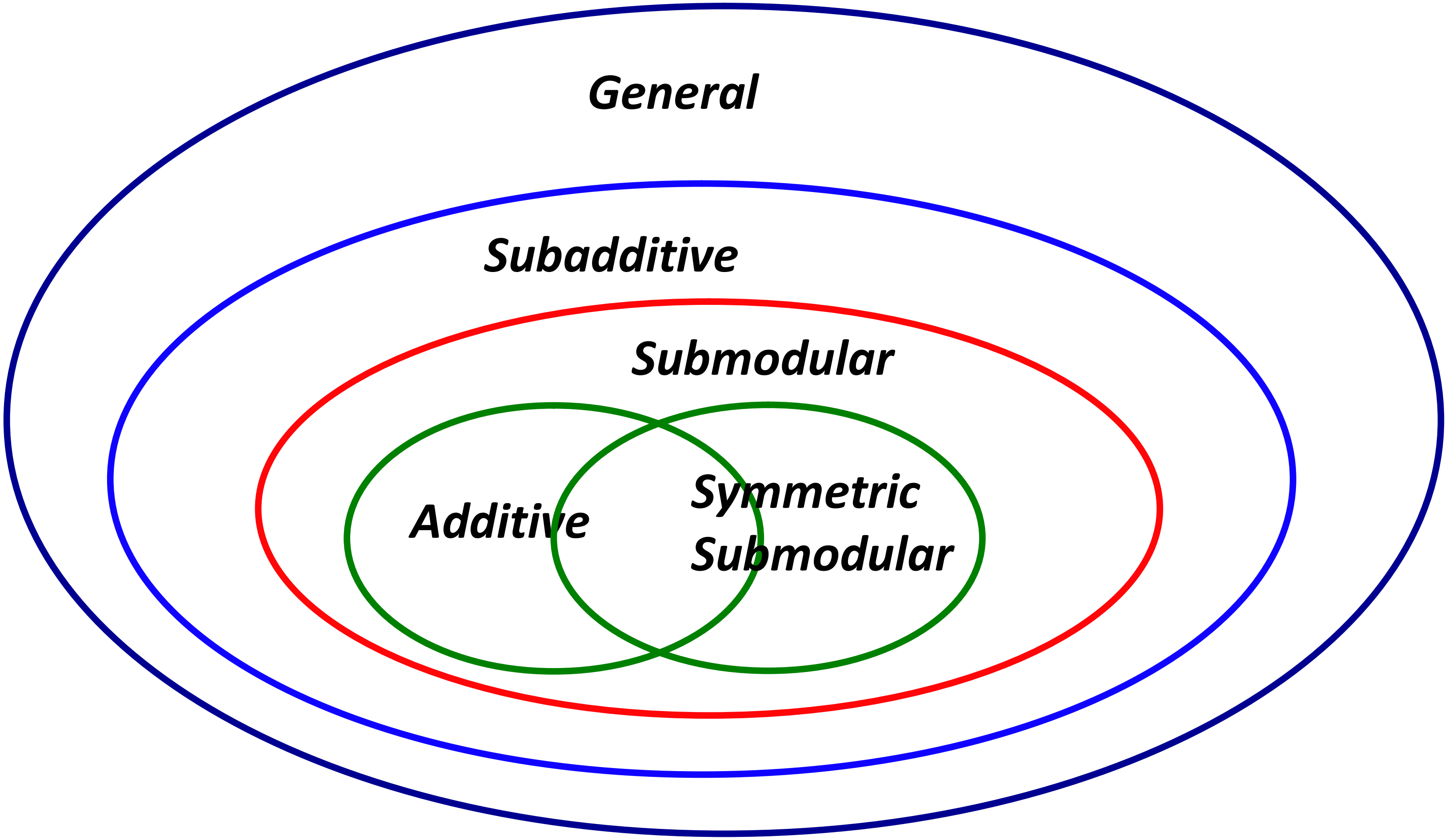
$$v_i(S \cup T) \geq v_i(S) + v_i(T)$$

•In this case, we have complementarity

•For example, the items may not have any value if they are sold on their own, but only when sold in bundles with other goods

- Single-minded bidders fall under this class

Relations between different classes of valuation functions



Social Welfare Maximization

- Need to define social welfare in this more general setting
- Definition: Let **S** = (S_1, S_2, ..., S_n) be an allocation of the items to the players, where S_i = subset assigned to player i. Then the social welfare derived from S is

$$SW(\mathbf{S}) = \Sigma_i \, v_i(S_i)$$

The SWM problem (Social Welfare Maximization):

Input: The valuation functions of the players (how?)

Output: Find an allocation $\mathbf{S}^*$ = (S_1 , S_2 , ..., S_n) that produces the highest possible social welfare:

$$SW(\mathbf{S}^*) \geq SW(\mathbf{S}) \text{ for any other allocation } \mathbf{S}$$

Social welfare maximization

Example with additive valuations

- 3 players, 4 items
- The input can be determined by a 3 x 4 array

48	**41**	11	0
35	10	**50**	5
45	20	10	**25**

- Optimal allocation: $S^* = (S_1, S_2, S_3) = (\{1, 2\}, \{3\}, \{4\})$
- Optimal social welfare: 48 + 41 + 50 + 25 = 164

Integer Programming Formulation

$$\max \sum_{i,S} x_{i,S} v_i(S)$$

$$\sum_{S} x_{i,S} \leq 1 \qquad \forall i \in [n]$$

$$\sum_{i,S: j \in S} x_{i,S} \leq 1 \qquad \forall j \in [m]$$

$$x_{i,S} \geq 0$$

$$\min \sum_{j \in [m]} p_j + \sum_{i \in [n]} u_i$$

$$u_i \geq v_i(S) - \sum_{j \in S} p_j \qquad \forall i, S$$

$$p_j \geq 0 \qquad \forall j \in [m]$$

$$u_i \geq 0 \qquad \forall i \in [n]$$

- p_j is the price of item *j* and u_i is the utility of bidder i

$$u_i = \max_S \{v_i(S) - p(S)\}$$

- Complementary slackness: in optimal solution (assuming integrality), each bidder gets a utility maximizing set and each item with positive price is allocated.
- Optimal solutions, if integral, correspond to equilibrium!

Walrasian (Competitive) Equilibrum

- **Competitive (Walrasian) equilibrium** is price vector **p** = (p_1, …, p_m) and allocation **S*** = (S_1, …, S_m) such that
  - $v_i(S_i) - p(S_i) \geq v_i(S) - p(S)$, for any subset S of items.
  - Every item j with $p_j > 0$ is allocated.

- Example: two bidders Alice and Bob, two items x and y.
  - Alice has value 2 for x, y and x+y, 0 for empty set.
  - Bob has value 4 for x+y and 0 for anything else.
  - $p_x = p_y = 2$, Alice nothing, Bob x+y is equilibrium.
  - If Bob had value 3 for x+y and 0 for anything else, Walrasian equilibrium does **not** exist!

Walrasian (Competitive) Equilibrum

- **Competitive (Walrasian) equilibrium** is price vector **p** = (p_1, …, p_m) and allocation $\mathbf{S}^*$ = (S_1, …, S_m) such that
  - $v_i(S_i) - p(S_i) \geq v_i(S) - p(S)$, for any subset S of items.
  - Every item j with $p_j > 0$ is allocated.
- **First Welfare Theorem:** (If exists,) Walrasian equilibrium maximizes social welfare, even among fractional solutions.

For any feasible (fractional) solution $x_{i,S}$, for any bidder i,

$$v_i(S_i) - \sum_{j \in S_i} p_j \geq \sum_S x_{i,S} \left(v_i(S) - \sum_{j \in S} p_j \right) \qquad (1)$$

by first condition and because $\sum_S x_{i,S} \leq 1$.

  - We sum up (1) and observe that sums of prices cancel out, because allocations must be disjoint.

Walrasian (Competitive) Equilibrum

- **Competitive (Walrasian) equilibrium** is price vector **p** = $(p_1, \ldots, p_m)$ and allocation $\mathbf{S}^* = (S_1, \ldots, S_m)$ such that
  - $v_i(S_i) - p(S_i) \geq v_i(S) - p(S)$, for any subset S of items.
  - Every item j with $p_j > 0$ is allocated.
- **Second Welfare Theorem:** If LP admits integral optimal solution, then Walrasian equilibrium exists.
  - Follows from complementary slackness conditions.
- LP admits integral optimal solution for **gross substitutes**.
  - When price for item increases, the demand for other items does not decrease.
  - Walrasian equilibrium computed by natural tatonnement process. [Kelso-Crawford, '82] Special case of discrete convexity!!!
  - http://www.inbaltalgam.com/slides/GS%20Tutorial%20Part%20I.pdf and http://www.inbaltalgam.com/slides/GS%20Tutorial%20Part%20II.pdf

Walrasian Tatonnement

- Demand correspondence:

$$D(v,p) = \{S \subseteq U : v(S) - p(S) \geq v(T) - p(T),\ \forall T \subseteq U\}$$

$$D_i(p) = \{S \subseteq U : v_i(S) - p(S) \geq v_i(T) - p(T),\ \forall T \subseteq U\}$$

An item-price ascending auction for substitutes valuations:

Initialization:

For every item $j \in M$, set $p_j \leftarrow 0$.

For every bidder i let $S_i \leftarrow \emptyset$.

Repeat

For each i, let D_i be the demand of i at the following prices:

p_j for $j \in S_i$ and $p_j + \epsilon$ for $j \notin S_i$.

If for all i $S_i = D_i$, exit the loop;

Find a bidder i with $S_i \neq D_i$ and update:

- For every item $j \in D_i \setminus S_i$, set $p_j \leftarrow p_j + \epsilon$
- $S_i \leftarrow D_i$
- For every bidder $k \neq i$, $S_k \leftarrow S_k \setminus D_i$

Finally: Output the allocation $S_1, \ldots, S_n$.

Mechanisms for Combinatorial Auctions

How do the players describe their valuations to auctioneer?

- For a general function, the bidder would need to specify $v_i(S)$, for every $S \subseteq M$ (2^m numbers, prohibitive!)
- Three approaches:
  1. Some functions can be described with a small number of parameters
     - E.g. additive or symmetric submodular (m parameters)
  2. The auctioneer can ask the bidders during the auction for their values on certain subsets of items
     - **Value queries.**
     - No need to know the entire function.
  3. The auctioneer computes prices and let the bidders decide on their utility maximizing set.
     - **Demand queries** – NP-hard to compute, in general.
     - No information about valuation is given to auctioneer.

Mechanisms for Combinatorial Auctions

- Truthful mechanisms for combinatorial auctions?
- Can we generalize the 2nd price auction when we have multiple items?
- We need to generalize:
  - The allocation algorithm: with 1 item, the winner was the highest bidder
    - multiple winners (with non-overlapping sets of goods), but monotonicity still necessary!
  - The payment rule: with 1 item, we offered a «discount» to the winner
    - Adjust the discount to the more general setting (and we also need a separate discount for each winner)

The VCG mechanism

- A generalization of the Vickrey auction
- Named after [Vickrey '61, Clarke '71, Groves '73]

1. **S*** = (S_1, S_2, …, S_n) social welfare maximizing allocation.
2. Allocation rule: For i=1, …, n, player i receives set S_i
3. Payment rule:
   - Payment of player i: $p_i = SW_{-i}^* - \sum_{j \neq i} v_j(S_j)$
     where SW_{-i}^* = optimal social welfare without player i
   - Every player pays the "externality" that his presence causes to the welfare of the others
   - Utility (value – payment) of player i: $u_i = SW^* - SW_{-i}^*$
   - Every player has utility equal to the increase in the social welfare due to his presence.

The VCG mechanism

In conclusion:

•Every player receives the items specified by the optimal allocation (w.r.t. the social welfare)

•His payment is determined by the declarations of the other players, just as in the Vickrey auction

Theorem: For any valuation functions, the VCG mechanism is truthful and maximizes the social welfare

Can we implement efficiently the VCG mechanism?

-Only when we can solve the SWM problem efficiently

The VCG Mechanism Truthfulness

Fix i and $\mathbf{b}_{-i}$. When the chosen outcome $\mathbf{x}(\mathbf{b})$ is ω^*, i's utility is

$$v_i(\omega^*) - p_i(\mathbf{b}) = \underbrace{\left[v_i(\omega^*) + \sum_{j \neq i} b_j(\omega^*) \right]}_{\text{(A)}} - \underbrace{\left[\max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega) \right]}_{\text{(B)}} .$$

- Part (B) is independent of i' bid b_i (truthfulness holds for any (B) that does not depend on b_i).
  - Part (B), a.k.a. Clarke pivot rule, ensures non-positive transfers (NPT) and individual rationality (IR).
- Bidding truthfully, i.e. $b_i = v_i$, allows the mechanism to maximize part (A), which is exactly what player i wants!
  - Players' incentives fully aligned with objective of mechanism!

Implementing the VCG mechanism

How to compute the allocation and the payment rule of VCG:

- It suffices to solve n+1 instances of the SWM problem
- 1 instance with all players present to determine the allocation of the items
- n more instances with a different player absent each time (SWM with n-1 out of the initial n players)
- Final complexity: O(n) · (complexity of SWM)

Implementing the VCG mechanism

Additive valuations

- Input: n x m matrix
- Solving SWM: Easy, greedy algorithm
  - For every item j: give it to the player with the highest value
- Implementing the payment rule of VCG:
  - Easy, solve n more times SWM with 1 player absent each time

Implementing the VCG mechanism

Example with additive valuations

- 3 players, 4 items

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- Optimal allocation: $S^* = (S_1, S_2, S_3) = (\{1, 2\}, \{3\}, \{4\})$
- Optimal social welfare: 48 + 41 + 50 + 25 = 164

Implementing the VCG mechanism

Example with additive valuations

•3 players, 4 items

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45	**20**	10	**25**

Payments:

- $p_1 = SW_{-1}^* - \Sigma_{j \neq 1} v_j(S_j) = 140 - (50+25) = 65$
- $p_2 = SW_{-2}^* - \Sigma_{j \neq 2} v_j(S_j) = 125 - (89+25) = 11$
- Similarly, $p_3 = 5$

Implementing the VCG mechanism

Additive valuations

- What if we run m independent Vickrey auctions for every item separately?
- We get the same result!
- It is due to the fact that we have additive valuations (hence, the values of different items for a player are not correlated)

Corollary:

For additive valuations, the VCG mechanism is equivalent to executing an independent Vickrey auction for each item

Implementing the VCG mechanism

Submodular functions?

Good news

Theorem: The VCG mechanism can be implemented in polynomial time for **symmetric** submodular valuations

- Greedy (wrt. marginal values) allocation is optimal.

Bad news

- For general submodular valuations, SWM is NP-complete
  - Reduction from Knapsack
- The same also holds for subadditive valuations

Implementing the VCG mechanism

Submodular functions?

[Lehmann, Lehmann, Nisan ’01]: greedy, 1/2-approximation

- Fix an ordering of the goods, 1, 2, ..., m
- For j = 1, ..., m
  - Let (S_1, S_2, ..., S_n) be the current allocation to the bidder
  - Allocate next good to the bidder with **currently highest marginal value** for this good
    - i.e., calculate $v_i(S_i \cup \{j\}) - v_i(S_i)$ for each player i
    - We measure how much extra welfare is derived by adding the good to the currently assigned bundle of a player

Implementing the VCG mechanism

Submodular functions?

- Further progress: ($1 - 1/e \approx 0.632$)-approximation with value queries [Vondrak '08]
- [Mirrokni, Schapira, Vondrak '08]: Better approximation would require exponentially many value queries.
- Unfortunately these algorithms cannot be combined with the VCG payment formula to obtain a truthful mechanism
- Open problem to derive a computationally efficient **truthful** mechanism for submodular valuations with the best possible approximation to the social welfare

Truthful Mechanisms for Subadditive Valuations

Value Queries [Dobzinski, Nisan, Schapira 05]:

1. Query each bidder for values of all singleton sets and U.
2. Find best "matching" allocation where each bidder gets at most one good (maximum bipartite matching).
   - Complete bipartite graph with agents on the left, goods on the right, and weight $v_i(\{ j \})$ on each { agent i, good j} edge.
3. Return best of maximum "matching" and max$\{ v_i(U) \}$

- Algorithm finds optimal over **subset** of feasible allocations, that includes only "matchings" and "winner-takes-all".
  - **Maximal-in-Range** (MiR) mechanisms: optimize over a predetermined subset of feasible solutions (a.k.a. "range").
  - Allocation is optimal-in-range: truthfulness with VCG payments!
  - Range chosen to guarantee good approximation and polynomial-time optimization.

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3. Return best of maximum "matching" and max{ $v_i(U)$ }

- Approximation ratio O(sqrt(m)) for subadditive valuations.
  - If most of OPT SW by "large sets" (cardinality $\geq$ sqrt(m), so at most sqrt(m) of them), max{ $v_i(U)$ } is sqrt(m)-approximation.
  - If most of OPT SW by "small sets" (cardinality < sqrt(m)) maximum "matching" is sqrt(m)-approximation (due to subadditivity and bound on cardinality).

Truthful Mechanisms for Subadditive Valuations

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3. Return best of maximum "matching" and max$\{ v_i(U) \}$

- **Theorem.** MiR algorithm above is truthful with VCG payments and achieves sqrt(m)-approximation for subadditive valuations.
- Maximal-in-Distributional Range gives sqrt(m)-approximation for CAs with **general** valuations [Lavi, Swamy '05]
  https://www.cs.princeton.edu/~smattw/Teaching/521fa17lec19.pdf
  https://www.math.uwaterloo.ca/~cswamy/papers/mechdeslp-journ.pdf

Linear Programming Relaxation of Social Welfare Maximization

$$\max \sum_{i,S} x_{i,S} v_i(S)$$

$$\sum_{S} x_{i,S} \leq 1 \qquad \forall i \in [n]$$

$$\sum_{i,S:j\in S} x_{i,S} \leq 1 \qquad \forall j \in [m]$$

$$x_{i,S} \geq 0$$

$$\min \sum_{j\in[m]} p_j + \sum_{i\in[n]} u_i$$

$$u_i \geq v_i(S) - \sum_{j\in S} p_j \qquad \forall i, S$$

$$p_j \geq 0 \qquad \forall j \in [m]$$

$$u_i \geq 0 \qquad \forall i \in [n]$$

- p_j is the price of item *j* and u_i is the utility of bidder i

$$u_i = \max_S \{v_i(S) - p(S)\}$$

$$D_i(U_i, p) = \{S \subseteq U_i : v_i(S) - p(S) \geq v_i(T) - p(T),\ \forall T \subseteq U_i\}$$

Truthful Mechanisms for Submodular Valuations

Demand Queries [Krysta, Vocking, '12]:

Algorithm 1. Overselling MPU algorithm
1 For each good $e \in U$ do $p_e^1 := p_0$.
2 For each bidder $i = 1, 2, \ldots, n$ do
3 Set $S_i := D_i(U_i, p^i)$, for a suitable $U_i \subseteq U$.
4 Update for each good $e \in S_i$: $p_e^{i+1} := p_e^i \cdot 2$

- Binary search in optimal prices of goods!
- **Truthful** because prices p_i do not depend on bidder i and demand queries.
- If $p_0 = \max\{ v_i(U) \} / (4m)$, Alg1 allocates $\leq \log_2(4m)+1$ copies of each good.
  - After allocating so many copies of good e, $p_e > \max\{ v_i(U) \}$ and no player can afford it anymore.

Truthful Mechanisms for Submodular Valuations

Lemma. p_e^* denotes final price of good e. Then,

$$\text{Alg} = \sum_{i=1}^{n} v_i(S_i) \geq \sum_{e \in U} p_e^* - m p_0$$

$$\sum_{i=1}^{n} v_i(S_i) \geq \sum_{i=1}^{n} \sum_{e \in S_i} p_e^i$$

$$= \sum_{i=1}^{n} \sum_{e \in S_i} 2^{\ell_e^i} p_0 \qquad \ell_e^i = \text{copies of } i \text{ sold before } i$$

$$= p_0 \sum_{e \in U} \sum_{k=1}^{\ell_e^* - 1} 2^k \qquad \ell_e^* = \text{copies } i \text{ sold in total}$$

$$= p_0 \sum_{e \in U} (2^{\ell_e^*} - 1)$$

$$= \sum_{e \in U} p_e^* - m p_0 = \sum_{e \in U} p_e^* - \frac{\text{OPT}}{4} \qquad p_0 = \frac{\max\{v_i(U)\}}{4m} \leq \frac{\text{OPT}}{4m}$$

Truthful Mechanisms for Submodular Valuations

- Approximation ratio: compare social welfare of Alg and OPT
  - Demand query ensures that:

$$\forall \, \text{player } i, \;\; v_i(S_i) \geq v_i(S_i^*) - \sum_{e \in S_i^*} p_e^*$$

  - Summing up and using Lemma:

$$\text{Alg} \geq \text{OPT} - \sum_{e \in U} p_e^* \geq \text{OPT} - \text{Alg} - \frac{\text{OPT}}{4}$$

  - We get **Alg ≥ 3OPT/8** (but with logarithmic "overselling").

Truthful Mechanisms for Submodular Valuations

Algorithm 1. Overselling MPU algorithm

For each good $e \in U$ do $p_e^1 := p_0$.
For each bidder $i = 1, 2, \ldots, n$ do
  Set $S_i := D_i(U_i, p^i)$, for a suitable $U_i \subseteq U$.
  Update for each good $e \in S_i$: $p_e^{i+1} := p_e^i \cdot 2$

- "Overselling" is fixed with oblivious rounding and sets U_i
  - U_i is the set of available goods at step i.
  - **After** the demand query $D_i(U_i, p^i)$ is answered, S_i is allocated with probability $1/\log_2(4m)$
  - Approximation ratio increases by factor $O(\log_2(4m))$ for submodular valuations.
- Demand-query truthful approximations extends to budgeted bidders and liquid welfare: $\text{LiquidValuation}_i(S) = \min\{ v_i(S), B_i \}$

Negative Cycles, Monotonicity and Truthfulness

- Consider allocation (and truthfulness) from viewpoint of single bidder (as in Myerson's Lemma, but multi-dimensional)
  - Fix allocation rule $\mathbf{x}$, other bids $\mathbf{b}_{-i}$ and payments $\mathbf{p}$.
  - Consider allocation $\mathbf{x}(b)$, payments $\mathbf{p}(b)$ and utility $v(\mathbf{x}(b)) - \mathbf{p}(b)$ of bidder i as functions of i's bid b and i's true valuation (a.k.a. type) v.
  - We want to characterize allocation rules $\mathbf{x}$ that allow for **truthful** payments $\mathbf{p}$ (similar to Myerson's Lemma).
  - Definition of truthfulness:

    $$v(\mathbf{x}(v)) - \mathbf{p}(v) \geq v(\mathbf{x}(b)) - \mathbf{p}(b), \text{ for all types } v, b$$
  - Focus on discrete domains (finite set of types), but everything generalizes to infinite (and continuous) domains.

Negative Cycles, Monotonicity and Truthfulness

- Let D set of all possible types.
- Correspondence graph G(D, E, w) is an edge-weighted complete directed graph on D.
  - Let b and b’ be two types / vertices and o = **x**(b) and o’ = **x**(b’) corresponding outcomes.
  - w(b, b’) = b(o) – b(o’) (and w(b’, b) = b’(o’) – b’(o)).
  - When true type b, how much bidder prefers o (outcome if he is truthful) to o’ (outcome if he misreports b’)
  - Payments p function of outcomes (only)!
- Allocation **x is truthful** (without payments!) iff w(b, b’) $\geq$ 0, for all edges (b, b’).

Negative Cycles, Monotonicity and Truthfulness

- Correspondence graph G(D, E, w).
  - Let b, b' be types and o = **x**(b), o' = **x**(b') outcomes.
  - w(b, b') = b(o) – b(o') (and w(b', b) = b'(o') – b'(o)).
- Allocation **x** admits **truthful** payments **p** : Outcomes -> R_+, if all edges (b, b') become non-negative after we apply **p**:

$$b(o) - p(o) \geq b(o') - p(o')$$

- Allocation **x** admits **truthful** payments **p** iff G(D, E, w) does not have negative cycles!
  - Truthful payments computed by Johnson's algorithm!
- If domain D is **convex**, allocation **x** admits **truthful** payments **p** iff G(D, E, w) does not have negative 2-cycles.
  - Weak monotonicity: b(o) – b'(o) ≥ b(o') – b'(o'), for all b, b' [Zaks, Yu '05]

Research questions on the implementation of truthful mechanisms

- Find special cases where SWM is solvable in polynomial time
- Design approximation algorithms for SWM for various types of valuation functions
- General problem with approximation algorithms: they cannot always be combined with some payment rule and get a truthful mechanism
- At the end, we need to understand how truthful mechanisms look like for multi-parameter environments, esp. when SWM is difficult