Algorithmic Game Theory: Introduction

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What is Game Theory?

Quote from M. Osborne: Game Theory aims to help us understand situations in which *decision makers* interact

- Goals:
  - Mathematical models for capturing the properties of such interactions
  - Prediction of behavior (given a model how should/would a *rational agent* act?)

- *Decision-makers:* humans, robots, computer programs, companies, political parties, etc
- *Rational:* when given a choice, the agent always chooses the option that yields the highest utility

Why Game Theory?

In a nutshell, game theory

- helps the designer of a system in understanding the strategic behavior of the participants, and in designing rules to align his\her incentives with the player's incentives
- helps a player in understanding the behavior of his competitors and in selecting a good strategy for himself
- Helps us decompose a complex decision-making problem into elementary decision dilemmas

Why Algorithmic Game Theory?

AGT is built on a fusion of ideas and models from computer science and game theory

- Many problems within game theory are inherently algorithmic
- Game theory only prescribes solutions but without telling us how to compute them
  - Hence, we need to design algorithms for effectively computing game theoretic concepts
  - Representative problem: Compute a Nash equilibrium in a game
- On the opposite side, many problems that arise within computer science are inherently game theoretic
  - Hence, we need to exploit models from game theory to capture such interactions
  - Representative problems: resource allocation among selfish entities, network routing protocols

Killer apps

- Sponsored search auctions (used by most search engines worldwide)
  - Make up a significant percentage of their revenue
- Spectrum auctions
  - For allocating spectrum telecom licences
- Matching programs
  - For matching doctors to hospitals, teachers to schools, etc (mostly in US and UK)
- Kidney exchange
  - For finding compatible donors for kidney transplants

Models of Games

Models of games

What is a game?

Any process where

- There are ≥ 2 entities
- All entities need to take some decision
- The final outcome and the utility of each player is determined by the choices of all players

Examples: board games, auctions, elections, ...

Models of games

Classes of games

- Cooperative or non-cooperative
- Simultaneous or sequential moves
- Repeated or not
- Finite or infinite
- Complete or incomplete information (presence of uncertainty)

Games in normal form

For most of this course, we will focus on games that are:

- Non-cooperative
  - The players do not communicate during play or do not form coalitions
- Complete information
  - The players are aware of the other players' preferences (but not of the decision they will take)
- Simultaneous moves
  - The players may not decide at the same time but at the moment where one players selects his action, he does not know and he cannot observe the other players' actions

Games in normal form

Definition: A game in normal form consists of

- A set of players $N = \{1, 2, \dots, n\}$
- For every player i, a set of available strategies S^i
- For every player i, a utility function $u_i: S^1 \times \dots \times S^n \rightarrow \mathbb{R}$

- Strategy profile: any vector in the form $(s_1, \dots, s_n)$, with $s_i \in S^i$
  - Every profile corresponds to an outcome of the game
  - The utility function describes the benefit/happiness that a player derives from the outcome of the game

2-player games in normal form

Consider a 2-player game with finite strategy sets

- $N = \{1, 2\}$
- $S^1 = \{s_1, \ldots, s_n\}$
- $S^2 = \{t_1, \ldots, t_m\}$
- Utility functions:
  $u_1: S^1 \times S^2 \to \mathbb{R}$, $u_2: S^1 \times S^2 \to \mathbb{R}$

- Possible strategy profiles:

$(s_1, t_1), (s_1, t_2), (s_1, t_3), \ldots, (s_1, t_m),$

$(s_2, t_1), (s_2, t_2), (s_2, t_3), \ldots, (s_2, t_m),$

...

$(s_n, t_1), (s_n, t_2), (s_n, t_3), \ldots, (s_n, t_m),$

2-player games in normal form

The utility function of each player can be described by a matrix of size n x m

- We can think of player 1 as having to select a row
- And of player 2 as having to select a column

- A finite 2-player game in normal form is defined by a pair of n x m matrices (A, B), where:
  - $A_{ij} = u_1(s_i, t_j)$, $B_{ij} = u_2(s_i, t_j)$
  - Player 1 is referred to as the row player
  - Player 2 is referred to as the column player

2-player games in normal form

Representation in matrix form:

For brevity, we will group together the values of the matrices A, B

$u_1(s_1, t_1), u_2(s_1, t_1)$	…, …	…, …	…, …	$u_1(s_1, t_m), u_2(s_1, t_m)$
$u_1(s_2, t_1), u_2(s_2, t_1)$	…, …	…, …	…, …	…, …
		$u_1(s_i, t_j), u_2(s_i, t_j)$	…, …	…, …
		…, …	…, …	…, …
…, …	…, …	…, …	…, …	$u_1(s_n, t_m), u_2(s_n, t_m)$

2-player games in normal form

Alternative representation:

We could use an ordering of all the outcome according to the preferences of each player

$>_1$: ordering of player 1

$>_2$: ordering of player 2

For example,

$(s_1, t_2) >_1 (s_2, t_3)$ means that player 1 prefers the outcome that results from the strategy profile (s_1, t_2) than the outcome from the profile (s_2, t_3)

- Possible issue when we use rankings: expressing ties in the utilities from different outcomes

Basic Examples of Games

Example 1: Prisoner's Dilemma

- Two suspects are being interrogated in separate cells for a crime they have committed
- If they do not confess, the police has evidence to condemn them for a more minor offence (1 year in jail for both)
- If they both confess, they will be sentenced to 3 years in jail each
- If only one confesses and the other denies, then the confesser is set free and the other suspect is sentenced to 4 years in jail
- The 2 suspects cannot communicate during interrogation

Example 1: Prisoner's Dilemma

- Set of players, N = {1, 2}
- Available strategies:
  - $S^1 = S^2$ = {Cooperate (C), Defect (D)}
- Possible outcomes
  - (C, C) = 1 year in jail for both
  - (C, D) = 4 years for pl. 1, pl. 2 is set free
  - (D, C) = pl. 1 is set free, 4 years for pl. 2
  - (D, D) = 3 years for both

Example 1: Prisoner's Dilemma

Preferences of the players:

- For player 1:

  (D, C) $>_1$ (C, C) $>_1$ (D, D) $>_1$ (C, D)

- For player 2:

  (C, D) $>_2$ (C, C) $>_2$ (D, D) $>_2$ (D, C)

- Representation in matrix form:
  - There are many equivalent ways
  - It suffices to choose utilities that are consistent with the rankings of the players
  - E.g. we could choose
    - $u_1(C, C) = 3$, $u_2(C, C) = 3$
    - $u_1(C, D) = 0$, $u_2(C, D) = 4$
    - $u_1(D, C) = 4$, $u_2(D, C) = 0$
    - $u_1(D, D) = 1$, $u_2(D, D) = 1$

Prisoner's Dilemma: Representation in matrix form

	C	D
C	3, 3	0, 4
D	4, 0	1, 1

- We could not have used the following form

3, 3	2, 4
4, 0	1, 1

here $u_1(C, D) > u_1(D, D)$

Prisoner’s Dilemma

- One of the most well studied games
- Extensive experimentation
- The game expresses one of the most fundamental dilemmas for the 2 players: To cooperate or not?
- This type of dilemma shows up in various scenarios and applications:
  - Joint project games
  - Duopoly model
  - Arms race

A Simple Duopoly Model

- 2 firms produce a product of similar quality (say 2 firms with sports shoes)
- Each of them needs to decide whether to ask for a high (H) or low (L) price
- Each firm would prefer as an ideal outcome that it sets a low price and the other form sets a high price
- Available strategies:
  - $S^1 = S^2$ = {High (H), Low (L)}
  - Preferences for pl. 1: (L, H) $>_1$ (L, L) $>_1$ (H, H) $>_1$ (H, L)
- This simple model of competition is equivalent to prisoner's dilemma!

Arms Race

- Popular during the cold war
- 2 countries (think of USA and Russia after the end of World War 2) want to decide if they will develop nuclear arms
- The ideal outcome for each country is to develop nuclear arms while the opponent abstains
- Available strategies:
  - $S^1 = S^2$ = {abstain from developing nuclear arms, develop nuclear arms}
  - Again, the preferences make the problem equivalent to prisoner's dilemma

Example 2: Bach or Stravinsky (BoS)

vs

- 2 players, a man and a woman
- 2 concerts of classical music, one dedicated to Bach, and one to Stravinsky
- The man prefers Bach, the woman prefers Stravinsky
- Both the man and the woman prefer to go somewhere together than go alone to a concert
- The dilemma here is not whether the players will cooperate but which concert to choose to go to

Example 2: Bach or Stravinsky (BoS)

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

- Any representation is acceptable as long as
  - $u_1(B, B) > u_1(S, S)$
  - $u_1(S, S) > u_1(S, B)$, $u_1(S, S) > u_1(B, S)$
  - Similarly for pl. 2
  - It is not important (for the time being) if $u_1(B, S) = u_1(S, B)$, it suffices that both are less than $u_1(S, S)$
- The game is also known as “Battle of the Sexes”

Example 3: The Hawk-Dove game

VS

- An example from Evolutionary Game Theory
- Two animal populations discover at the same time a valley with food resources
- The two populations can share the valley without attacking each other
- Otherwise, they can also choose to attack each other

Example 3: The Hawk-Dove game

		
	(2, 2)	(0, 4)
	(4, 0)	(-1, -1)

- $S^1 = S^2 =$ {Friendly dove (D), Aggressive hawk(H)}
- It suffices to have any matrix representation where
  - $u_1(H, D) > u_1(D, D) > u_1(D, H) > u_1(H, H)$
- Variations and generalizations of this game help in understanding the evolution of animal populations under competition

Example 4: Matching Pennies

vs

- Two players hold a coin each
- Each player should decide if he will display Heads or Tails
- Player 1 wins if they show the same side of the coin
- Player 2 wins if they show different sides
- Known also as the penalty-kick game
  - Players: the goalkeeper and the striker of the opposite team
  - Available strategies: for the goalkeeper, choose a side to fall, for the striker, choose towards which side to shoot
  - The goalkeeper wins if they choose the same side
  - The striker wins if they choose different sides

Example 4: Matching Pennies

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

- $S^1 = S^2 = \{H, T\}$
- This is an example of a 0-sum game, because
  - $u_1(s, t) + u_2(s, t) = 0$, for every profile (s, t)

Example 4: Matching Pennies

- Rock-Paper-Scissors: An extension of Matching Pennies
- $S^1 = S^2 = \{R, S, P\}$
- Again a 0-sum game

	R	S	P
R	0, 0	1, -1	-1, 1
S	-1, 1	0, 0	1, -1
P	1, -1	-1, 1	0, 0