THE ARCHITECTURE OF MATHEMATICS*

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1. Mathematic or mathematics? To present a view of the entire field of mathematical science as it exists,—this is an enterprise which presents, at first sight, almost insurmountable difficulties, on account of the extent and the varied character of the subject. As is the case in all other sciences, the number of mathematicians and the number of works devoted to mathematics have greatly increased since the end of the 19th century. The memoirs in pure mathematics published in the world during a normal year cover several thousands of pages. Of course, not all of this material is of equal value; but, after full allowance has been made for the unavoidable tares, it remains true nevertheless that mathematical science is enriched each year by a mass of new results, that it spreads and branches out steadily into theories, which are subjected to modifications based on new foundations, compared and combined with one another. No mathematician, even were he to devote all his time to the task, would be able to follow all the details of this development. Many mathematicians take up quarters in a corner of the domain of mathematics, which they do not intend to leave; not only do they ignore almost completely what does not concern their special field, but they are unable to understand the language and the terminology used by colleagues who are working in a corner remote from their own. Even among those who have the widest training, there are none who do not feel lost in certain regions of the immense world of mathematics; those who, like Poincaré or Hilbert, put the seal of their genius on almost every domain, constitute a very great exception even among the men of greatest accomplishment.

It must therefore be out of the question to give to the uninitiated an exact picture of that which the mathematicians themselves can not conceive in its totality. Nevertheless it is legitimate to ask whether this exuberant proliferation makes for the development of a strongly constructed organism, acquiring ever greater cohesion and unity with its new growths, or whether it is the external manifestation of a tendency towards a progressive splintering, inherent in the very nature of mathematics, whether the domain of mathematics is not becoming a tower of Babel, in which autonomous disciplines are being more and more widely separated from one another, not only in their aims, but also in their methods and even in their language. In other words, do we have today a mathematic or do we have several mathematics?

Although this question is perhaps of greater urgency now than ever before, it is by no means a new one; it has been asked almost from the very beginning of mathematical science. Indeed, quite apart from applied mathematics, there has

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always existed a dualism between the origins of geometry and of arithmetic (certainly in their elementary aspects), since the latter was at the start a science of discrete magnitude, while the former has always been a science of continuous extent; these two aspects have brought about two points of view which have been in opposition to each other since the discovery of irrationals. Indeed, it is exactly this discovery which defeated the first attempt to unify the science, *viz.*, the arithmetization of the Pythagoreans ("everything is number").

It would carry us too far if we were to attempt to follow the vicissitudes of the unitary conception of mathematics from the period of Pythagoras to the present time. Moreover this task would suit a philosopher better than a mathematician; for it is a common characteristic of the various attempts to integrate the whole of mathematics into a coherent whole—whether we think of Plato, of Descartes or of Leibnitz, of arithmetization, or of the logistics of the 19th century—that they have all been made in connection with a philosophical system, more or less wide in scope; always starting from *a priori* views concerning the relations of mathematics with the twofold universe of the external world and the world of thought. We can do no better on this point than to refer the reader to the historical and critical study of L. Brunschvicg [1]. Our task is a more modest and a less extensive one; we shall not undertake to examine the relations of mathematics to reality or to the great categories of thought; we intend to remain within the field of mathematics and we shall look for an answer to the question which we have raised, by analyzing the procedures of mathematics themselves.

2. Logical formalism and axiomatic method. After the more or less evident bankruptcy of the different systems, to which we have referred above, it looked, at the beginning of the present century as if the attempt had just about been abandoned to conceive of mathematics as a science characterized by a definitely specified purpose and method; instead there was a tendency to look upon mathematics as "a collection of disciplines based on particular, exactly specified concepts," interrelated by "a thousand roads of communication," allowing the methods of any one of these disciplines to fertilize one or more of the others [1, page 447]. Today, we believe however that the internal evolution of mathematical science has, in spite of appearance, brought about a closer unity among its different parts, so as to create something like a central nucleus that is more coherent than it has ever been. The essential aspect of this evolution has been the systematic study of the relations existing between different mathematical théories, and which has led to what is generally known as the "axiomatic method."

The words "formalism" and "formalistic method" are also often used; but it is important to be on one's guard from the start against the confusion which may be caused by the use of these ill-defined words, and which is but too frequently made use of by the opponents of the axiomatic method. Everyone knows that superficially mathematics appears as this "long chain of reasons" of which Descartes spoke; every mathematical theory is a concatenation of propositions, each one derived from the preceding ones in conformity with the rules of a logical system, which is essentially the one codified, since the time of Aristotle, under the name of "formal logic," conveniently adapted to the particular aims of the mathematician. It is therefore a meaningless truism to say that this "deductive reasoning" is a unifying principle for mathematics. So superficial a remark can certainly not account for the evident complexity of different mathematical theories, not any more than one could, for example, unite physics and biology into a single science on the ground that both use the experimental method. The method of reasoning by means of chains of syllogisms is nothing but a transforming mechanism, applicable just as well to one set of premises as to another; it could not serve therefore to characterize these premises. In other words, it is the external form which the mathematician gives to his thought, the vehicle which makes it accessible to others,* in short, the language suited to mathematics; this is all, no further significance should be attached to it. To lay down the rules of this language, to set up its vocabulary and to clarify its syntax, all that is indeed extremely useful; indeed this constitutes one aspect of the axiomatic method, the one that can properly be called logical formalism (or "logistics" as it is sometimes called). But we emphasize that it is but one aspect of this method, indeed the least interesting one.

What the axiomatic method sets as its essential aim, is exactly that which logical formalism by itself can not supply, namely the profound intelligibility of mathematics. Just as the experimental method starts from the *a priori* belief in the permanence of natural laws, so the axiomatic method has its cornerstone in the conviction that, not only is mathematics not a randomly developing concatenation of syllogisms, but neither is it a collection of more or less "astute" tricks, arrived at by lucky combinations, in which purely technical cleverness wins the day. Where the superficial observer sees only two, or several, quite distinct theories, lending one another "unexpected support" [1, page 446] through the intervention of a mathematician of genius, the axiomatic method teaches us to look for the deep-lying reasons for such a discovery, to find the common ideas of these theories, buried under the accumulation of details properly belonging to each of them, to bring these ideas forward and to put them in their proper light.

3. The notion of structure. In what form can this be done? It is here that the axiomatic method comes closest to the experimental method. Like the latter drawing its strength from the source of Cartesianism, it will "divide the difficulties in order to overcome them better." It will try, in the demonstrations of a theory, to separate out the principal mainsprings of its arguments; then, taking each of these separately and formulating it in abstract form, it will develop

* Indeed every mathematician knows that a proof has not really been "understood" if one has done nothing more than verifying step by step the correctness of the deductions of which it is composed, and has not tried to gain a clear insight into the ideas which have led to the construction of this particular chain of deductions in preference to every other one.

the consequences which follow from it alone. Returning after that to the theory under consideration, it will recombine the component elements, which had previously been separated out, and it will inquire how these different components influence one another. There is indeed nothing new in this classical going to-and-fro between analysis and synthesis; the originality of the method lies entirely in the way in which it is applied.

In order to illustrate the procedure which we have just sketched, by an example, we shall take one of the oldest (and also one of the simplest) of axiomatic theories, *viz.* that of the "abstract groups." Let us consider for example, the three following operations: 1. the addition of real numbers, their sum (positive negative or zero) being defined in the usual manner; 2. the multiplication of integers "modulo a prime number p," (where the elements under consideration are the whole numbers $1, 2, \cdots, p-1$) and the "product" of two of these numbers is, by agreement, defined as the remainder of the division of their usual product by p; 3. the "composition" of displacements in three-dimensional Euclidean space, the "resultant" (or "product") of two displacements S, T (taken in this order) being defined as the displacement obtained by carrying out first the displacement T and then the displacement S. In each of these three theories, one makes correspond, by means of a procedure defined for each theory, to two elements x, y (taken in that order) of the set under consideration (in the first case the set of real numbers, in the second the set of numbers $1, 2, \cdots, p-1$, in the third the set of all displacements) a well-determined third element; we shall agree to designate this third element in all three cases by $x\tau y$ (this will be the sum of x and y if x and y are real numbers, their product "modulo p" if they are integers $\leqq p-1$, their resultant if they are displacements). If we now examine the various properties of this "operation" in each of the three theories, we discover a remarkable parallelism; but, in each of the separate theories, the properties are interconnected, and an analysis of their logical connections leads us to select a small number of them which are independent (*i.e.*, none of them is a logical consequence of all the others). For example,* one can take the three following, which we shall express by means of our symbolic notation, common to the three theories, but which it would be very easy to translate into the particular language of each of them:

(a) For all elements x, y, z, one has $x\tau(y\tau z) = (x\tau y)\tau z$ ("associativity" of the operation $x\tau y$);

(b) There exists an element e, such that for every element x, one has $e\tau x = x\tau e = x$ (for the addition of real numbers, it is the number 0; for multiplication "modulo p," it is the number 1; for the composition of displacements, it is the "identical" displacement, which leaves every point of space fixed);

(c) Corresponding to every element x, there exists an element x' such that $x\tau x' = x'\tau x = e$ (for the addition of real numbers x' is the number $-x$; for the

* There is nothing absolute in this choice; several systems of axioms are known which are "equivalent" to the one which we are stating explicitly, the axioms of each of these systems being logical consequences of the axioms of any other one.

composition of displacements, x' is the "inverse" displacement of x, *i.e.* the displacement which replaces each point that had been displaced by x to its original position; for multiplication "modulo p," the existence of x' follows from a very simple arithmetic argument.*

It follows then that the properties which can be expressed in the same way in the three theories, by means of the common notation, are consequences of the three preceding ones. Let us try to show, for example that from $x\tau y = x\tau z$ follows $y = z$; one could do this in each of the theories by a reasoning peculiar to it. But, we can proceed as follows by a method that is applicable in all cases: from the relation $x\tau y = x\tau z$ we derive (x' having the meaning which was defined above) $x'\tau(x\tau y) = x'\tau(x\tau z)$; thence by applying (a), $(x'\tau x)\tau y = (x'\tau x)\tau z$; by means of (c), this relation takes the form $e\tau y = e\tau z$, and finally, by applying (b), $y = z$, which was to be proved. In this reasoning the nature of the elements x, y, z under consideration has been left completely out of account; we have not been concerned to know whether they are real numbers, or integers $\leqq p-1$, or displacements; the only premise that was of importance was that the operation $x\tau y$ on these elements has the properties (a), (b), and (c). Even if it were only to avoid irksome repetitions, it is readily seen that it would be convenient to develop once and for all the logical consequences of the three properties (a), (b), (c) only. For linguistic convenience, it is of course desirable to adopt a common terminology for the three sets. One says that a set in which an operation $x\tau y$ has been defined which has the three properties (a), (b), (c) is provided with a group structure (or, briefly, that it is a group); the properties (a), (b), (c) are called the axioms of** the group structures, and the development of their consequences constitutes setting up the axiomatic theory of groups.

It can now be made clear what is to be understood, in general, by a mathematical structure. The common character of the different concepts designated by this generic name, is that they can be applied to sets of elements whose nature† has not been specified; to define a structure, one takes as given one or

* We observe that the remainders left when the numbers $x, x^2, \cdots, x^n, \cdots$ are divided by p, can not all be distinct; by expressing the fact that two of these remainders are equal, one shows easily that a power x^m of x exists which has a remainder equal to 1; if now x' is the remainder of the division of x^{m-1} by p, we conclude that the product "modulo p" of x and x' is equal to 1.

** It goes without saying that there is no longer any connection between this interpretation of the word "axiom" and its traditional meaning of "evident truth."

† We take here a naive point of view and do not deal with the thorny questions, half philosophical, half mathematical, raised by the problem of the "nature" of the mathematical "beings" or "objects." Suffice it to say that the axiomatic studies of the nineteenth and twentieth centuries have gradually replaced the initial pluralism of the mental representation of these "beings"—thought of at first as ideal "abstractions" of sense experiences and retaining all their heterogeneity—by an unitary concept, gradually reducing all the mathematical notions, first to the concept of the natural number and then, in a second stage, to the notion of set. This latter concept, considered for a long time as "primitive" and "undefinable," has been the object of endless polemics, as a result of its extremely general character and on account of the very vague type of mental representation which it calls forth; the difficulties did not disappear until the notion of set itself disappeared (and with it all the metaphysical pseudo-problems concerning mathematical "beings" in the light of the recent work on logical formalism. From this new point of view, mathematical

several relations, into which these elements enter* (in the case of groups, this was the relation $z=x\tau y$ between three arbitrary elements); then one postulates that the given relation, or relations, satisfy certain conditions (which are explicitly stated and which are the axioms of the structure under consideration.)† To set up the axiomatic theory of a given structure, amounts to the deduction of the logical consequences of the axioms of the structure, excluding every other hypothesis on the elements under consideration (in particular, every hypotheses as to their own nature).

4. The great types of structures. The relations which form the starting point for the definition of a structure can be of very different characters. The one which occurs in the group structure is what one calls a "law of composition," *i.e.*, a relation between three elements which determines the third uniquely as a function of the first two. When the relations which enter the definition of a structure are "laws of composition," the corresponding structure is called an algebraic structure (for example, a field structure is defined by two laws of composition, with suitable axioms: the addition and multiplication of real numbers define a field structure on the set of these numbers).

Another important type is furnished by the structures defined by an order relation; this is a relation between two elements x, y which is expressed most frequently in the form "x is at most equal to y," and which we shall represent in general by $x\mathrm{R}y$. It is not at all supposed here that it determines one of the two elements x, y uniquely as a function of the other; the axioms to which it is subjected are the following: (a) for every x we have $x\mathrm{R}x$; (b) from the relations $x\mathrm{R}y$ and $y\mathrm{R}x$ follows $x=y$; (c) the relations $x\mathrm{R}y$ and $y\mathrm{R}z$ have as a consequence $x\mathrm{R}z$. An obvious example of a set with a structure of this kind is the set of integers (or that of real numbers), when the symbol R is replaced by the symbol $\leqq$. But it must be observed that we have not included among the axioms the following property, which seems to be inseparable from the popular notion of "order," "for every pair of elements x and y, either $x\mathrm{R}y$ or $y\mathrm{R}x$ holds." In other words, the case in which x and y are incomparable is not excluded. This may seem paradoxical at first sight, but it is easy to give examples of very important order structures, in which such a phenomenon appears. This is what happens when X and Y denote parts of the same set and the relation $X\mathrm{R}Y$ is interpreted to mean "X is contained in Y"; again when x and y are positive integers and $x\mathrm{R}y$ means

structures become, properly speaking, the only "objects" of mathematics. The reader will find fuller developments of this point in articles by J. Dieudonné [2] and H. Cartan [3].

* In effect, this definition of structures is not sufficiently general for the needs of mathematics; it is also necessary to consider the case in which the relations which define a structure hold not between elements of the set under consideration, but also between parts of this set and even, more generally, between elements of sets of still higher "degree" in the terminology of the "hierarchy of types." For further details on this point, see [4].

† Strictly speaking, one should, in the case of groups, count among the axioms, besides properties (a), (b), (c) stated above, the fact that the relation $z=x\tau y$ determines one and only one z when x and y are given; one usually considers this property as tacitly implied by the form in which the relation is written.

"x divides y"; also if $f(x)$ and $g(x)$ are real-valued functions defined on an interval $a \leqq x \leqq b$, while $f(x)\mathrm{R}g(x)$ is interpreted to mean "for every $x, f(x) \leqq g(x)$." These examples also give an indication of the great variety of domains in which order structures appear and thus point to the interest attached to their study.

We want to say a few words about a third large type of structures, *viz.* topological structures (or topologies): they furnish an abstract mathematical formulation of the intuitive concepts of neighborhood, limit and continuity, to which we are led by our idea of space. The degree of abstraction required for the formulation of the axioms of such a structure is decidedly greater than it was in the preceding examples; the character of the present article makes it necessary to refer interested readers to special treatises. See, for example, [5].

5. The standardization of mathematical technique. We have probably said enough to enable the reader to form a fairly accurate idea of the axiomatic method. It should be clear from what precedes that its most striking feature is to effect a considerable economy of thought. The "structures" are tools for the mathematician; as soon as he has recognized among the elements, which he is studying, relations which satisfy the axioms of a known type, he has at his disposal immediately the entire arsenal of general theorems which belong to the structures of that type. Previously, on the other hand, he was obliged to forge for himself the means of attack on his problems; their power depended on his personal talents and they were often loaded down with restrictive hypotheses, resulting from the peculiarities of the problem that was being studied. One could say that the axiomatic method is nothing but the "Taylor system" for mathematics.

This is however, a very poor analogy; the mathematician does not work like a machine, nor as the workingman on a moving belt; we can not over-emphasize the fundamental role played in his research by a special intuition,* which is not the popular sense-intuition, but rather a kind of direct divination (ahead of all reasoning) of the normal behavior, which he seems to have the right to expect of mathematical beings, with whom a long acquaintance has made him as familiar as with the beings of the real world. Now, each structure carries with it its own language, freighted with special intuitive references derived from the theories from which the axiomatic analysis described above has derived the structure. And, for the research worker who suddenly discovers this structure in the phenomena which he is studying, it is like a sudden modulation which orients at one stroke in an unexpected direction the intuitive course of his thought, and which illumines with a new light the mathematical landscape in which he is moving about. Let us think—to take an old example—of the progress made at the beginning of the nineteenth century by the geometric representation of imaginaries. From our point of view, this amounted to discovering in the set of complex numbers a well-known topological structure, that of the Euclidean plane, with all the possibilities for applications which this in-

* Like all intuitions, this one also is frequently wrong.

volved; in the hands of Gauss, Abel, Cauchy and Riemann, it gave new life to analysis in less than a century. Such examples have occurred repeatedly during the last fifty years; Hilbert space, and more generally, functional spaces, establishing topological structures in sets whose elements are no longer points, but functions; the theory of the Hensel p-adic numbers, where, in a still more astounding way, topology invades a region which had been until then the domain *par excellence* of the discrete, of the discontinuous, *viz.* the set of whole numbers; Haar measure, which enlarged enormously the field of application of the concept of integral, and made possible a very profound analysis of the properties of continuous groups;—all of these are decisive instances of mathematical progress, of turning points at which a stroke of genius brought about a new orientation of a theory, by revealing the existence in it of a structure which did not *a priori* seem to play a part in it.

What all this amounts to is that mathematics has less than ever been reduced to a purely mechanical game of isolated formulas; more than ever does intuition dominate in the genesis of discoveries. But henceforth, it possesses the powerful tools furnished by the theory of the great types of structures; in a single view, it sweeps over immense domains, now unified by the axiomatic method, but which were formerly in a completely chaotic state.

6. A general survey. Let us now try, guided by the axiomatic concept, to look over the whole of the mathematical universe. It is clear that we shall no longer recognize the traditional order of things, which, just like the first nomenclatures of animal species, restricted itself to placing side by side the theories which showed greatest external similarity. In place of the sharply bounded compartments of algebra, of analysis, of the theory of numbers, and of geometry, we shall see, for example, that the theory of prime numbers is a close neighbor of the theory of algebraic curves, or, that Euclidean geometry borders on the theory of integral equations. The organizing principle will be the concept of a hierarchy of structures, going from the simple to the complex, from the general to the particular.

At the center of our universe are found the great types of structures, of which the principal ones were mentioned above; they might be called the mother-structures. A considerable diversity exists in each of these types; one has to distinguish between the most general structure of the type under consideration, with the smallest number of axioms, and those which are obtained by enriching the type with supplementary axioms, from each of which comes a harvest of new consequences. Thus, the theory of groups contains, beyond the general conclusions valid for all groups and depending only on the axioms enunciated above, a particular theory of finite groups (obtained by adding the axiom that the number of elements of the group is finite), a particular theory of abelian groups (in which $x\tau y = y\tau x$ for every x and y), as well as a theory of finite abelian groups (where these two axioms are supposed to hold simultaneously). Similarly, in the theory of ordered sets, one notices in particular those sets (as for example, the set of integers, or of real numbers) in which any two elements are comparable, and which are called totally ordered. Among the latter, further attention is given to the sets which are called well-ordered (in which, as in the set of integers greater than 0, every subset has a "least element"). There is an analogous gradation among topological structures.

Beyond this first nucleus, appear the structures which might be called multiple structures. They involve two or more of the great mother-structures simultaneously not in simple juxtaposition (which would not produce anything new), but combined organically by one or more axioms which set up a connection between them. Thus, one has topological algebra. This is a study of structures in which occur at the same time, one or more laws of composition and a topology, connected by the condition that the algebraic operations be (for the topology under consideration) continuous functions of the elements on which they operate. Not less important is algebraic topology, in which certain sets of points in space, defined by topological properties (simplexes, cycles, etc.) are themselves taken as elements on which laws of composition operate. The combination of order structures and algebraic structures is also fertile in results, leading, in one direction to the theory of divisibility and of ideals, and in another to integration and to the "spectral theory" or operators, in which topology also joins in.

Farther along we come finally to the theories properly called particular. In these the elements of the sets under consideration, which, in the general structures have remained entirely indeterminate, obtain a more definitely characterized individuality. At this point we merge with the theories of classical mathematics, the analysis of functions of a real or complex variable, differential geometry, algebraic geometry, theory of numbers. But they have no longer their former autonomy; they have become crossroads, where several more general mathematical structures meet and react upon one another.

To maintain a correct perspective, we must at once add to this rapid sketch, the remark that it has to be looked upon as only a very rough approximation of the actual state of mathematics, as it exists; the sketch is schematic, and idealized as well as frozen.

Schematic—because in the actual procedures, things do not happen in as simple and as systematic a manner as has been described above. There occur, among other things, unexpected reverse movements, in which a specialized theory, such as the theory of real numbers, lends indispensable aid in the construction of a general theory like topology or integration.

Idealized—because it is far from true that in all fields of mathematics, the role of each of the great structures is clearly recognized and marked off; in certain theories (for example in the theory of numbers), there remain numerous isolated results, which it has thus far not been possible to classify, nor to connect in a satisfactory way with known structures.

Finally *frozen*,—for nothing is farther from the axiomatic method than a static conception of the science. We do not want to lead the reader to think that we claim to have traced out a definitive state of the science. The structures are not immutable, neither in number nor in their essential contents. It is quite possible that the future development of mathematics may increase the number of fundamental structures, revealing the fruitfulness of new axioms, or of new combinations of axioms. We can look forward to important progress from the invention of structures, by considering the progress which has resulted from actually known structures. On the other hand, these are by no means finished edifices; it would indeed be very surprising if all the essence had already been extracted from their principles. Thus, with these indispensable qualifications, we can become better aware of the internal life of mathematics, of its unity as well as of its diversity. It is like a big city, whose outlying districts and suburbs encroach incessantly, and in a somewhat chaotic manner, on the surrounding country, while the center is rebuilt from time to time, each time in accordance with a more clearly conceived plan and a more majestic order, tearing down the old sections with their labyrinths of alleys, and projecting towards the periphery new avenues, more direct, broader and more commodious.

7. Return to the past and conclusion. The concept which we have tried to present in the above paragraphs, was not formed all at once; rather is it a stage in an evolution, which has been in progress for more than a half-century, and which has not escaped serious opposition, among philosophers as well as among mathematicians themselves. Many of the latter have been unwilling for a long time to see in axiomatics anything else than futile logical hairsplitting not capable of fructifying any theory whatever. This critical attitude can probably be accounted for by a purely historical accident. The first axiomatic treatments and those which caused the greatest stir (those of arithmetic by Dedekind and Peano, those of Euclidean geometry by Hilbert) dealt with univalent theories, *i.e.*, theories which are entirely determined by their complete system of axioms; for this reason they could not be applied to any theory except the one from which they had been extracted (quite contrary to what we have seen, for instance, for the theory of groups). If the same had been true for all other structures, the reproach of sterility brought against the axiomatic method, would have been fully justified.* But the further development of the method has revealed its power; and the repugnance which it still meets here and there, can only be explained by the natural difficulty of the mind to admit, in dealing with a concrete problem, that a form of intuition, which is not suggested directly by the given elements (and which often can be arrived at only by a higher and frequently difficult stage of abstraction), can turn out to be equally fruitful.

As concerns the objections of the philosophers, they are related to a domain, on which for reasons of inadequate competence we must guard ourselves from

* There also occurred, especially at the beginning of axiomatics, a whole crop of monster-structures, entirely without applications; their sole merit was that of showing the exact bearing of each axiom, by observing what happened if one omitted or changed it. There was of course a temptation to conclude that these were the only results that could be expected from the axiomatic method.

entering; the great problem of the relations between the empirical world and the mathematical world.* That there is an intimate connection between experimental phenomena and mathematical structures, seems to be fully confirmed in the most unexpected manner by the recent discoveries of contemporary physics. But we are completely ignorant as to the underlying reasons for this fact (supposing that one could indeed attribute a meaning to these words) and we shall perhaps always remain ignorant of them. There certainly is one observation which might lead the philosophers to greater circumspection on this point in the future: before the revolutionary developments of modern physics, a great deal of effort was spent on trying to derive mathematics from experimental truths, especially from immediate space intuitions. But, on the one hand, quantum physics has shown that this macroscopic intuition of reality covered microscopic phenomena of a totally different nature, connected with fields of mathematics which had certainly not been thought of for the purpose of applications to experimental science. And, on the other hand, the axiomatic method has shown that the "truths" from which it was hoped to develop mathematics, were but special aspects of general concepts, whose significance was not limited to these domains. Hence it turned out, after all was said and done, that this intimate connection, of which we were asked to admire the harmonious inner necessity, was nothing more than a fortuitous contact of two disciplines whose real connections are much more deeply hidden than could have been supposed *a priori*.

From the axiomatic point of view, mathematics appears thus as a storehouse of abstract forms—the mathematical structures; and it so happens—without our knowing why—that certain aspects of empirical reality fit themselves into these forms, as if through a kind of preadaptation. Of course, it can not be denied that most of these forms had originally a very definite intuitive content; but, it is exactly by deliberately throwing out this content, that it has been possible to give these forms all the power which they were capable of displaying and to prepare them for new interpretations and for the development of their full power.

It is only in this sense of the word "form" that one can call the axiomatic method a "formalism." The unity which it gives to mathematics is not the armor of formal logic, the unity of a lifeless skeleton; it is the nutritive fluid of an organism at the height of its development, the supple and fertile research instrument to which all the great mathematical thinkers since Gauss have contributed, all those who, in the words of Lejeune-Dirichlet, have always labored to "substitute ideas for calculations."

* We do not consider here the objections which have arisen from the application of the rules of formal logic to the reasoning in axiomatic theories; these are connected with logical difficulties encountered in the theory of sets. Suffice it to point out that these difficulties can be overcome in a way which leaves neither the slightest qualms nor any doubt as to the correctness of the reasoning; [2] and [3] are valuable references for this point.

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THE GEOMETRIES OF THE THERMAL AND GRAVITATIONAL FIELDS*

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1. Introduction. In physics, geometry is—or should be—what you measure it. Deviations from the geometry or kinematics you expect to find, on *a priori* or other grounds, may lead to the discovery or better understanding of some hitherto overlooked element in the realm under examination. To such deviations may be attributed the discovery by the ancients of the curvature of the earth's surface, by Kepler of the laws of planetary motion, and by Einstein of the special and general theories of relativity. Perhaps the most spectacular in its impact on the modern mind, because we are still immersed in the intellectual milieu in which it arose, is Einstein's geometrization of the universal force of gravitation. Here the recognition of the appropriateness of a certain Riemannian kinematics made possible the incorporation of gravitation into the geometrical structure of space-time, and by identification of inertial and gravitational mass made tautological the law which states their equivalence.

To illustrate the relationship between geometry and physical law, I have elected here first to discuss in some detail the problem of the geometry exhibited by a "hot plate." We shall find, as could be foreseen, that the geometry disclosed by measurement depends not on the state of the plate alone, but also on the method of measurement, *e.g.*, whether by purely optical means or by the use of measuring rods which are brought into thermal equilibrium with the plate before being read. The arbitrariness (within, however, fairly closely prescribed limits) of the geometry thus found will support us in the common-sense view that the plate is actually flat, and that the observed deviations are to be attributed to quite well understood forces (thermal stresses) which act differ-

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