COMPLEX ANALYSIS
WORKSHEET 4
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1. Compute

$$\max\{|f(z)| : z \in K\}, \quad \min\{|f(z)| : z \in K\}$$

and the points where the above max and min are attained, in each of the following cases:
(i) $f(z) = |z^2 + 3z - 1|, \quad K = \{z \in \mathbb{C} : |z| \leq 1\}$.
(ii) $f(z) = e^{z^2}, \quad K$ is the closed bounded domain with boundary the triangle with vertices $0, \ -1, \ 1+i$.

2. Consider a holomorphic function f on an open set $U \subseteq \mathbb{C}$ which contains the closed annulus

$$\Delta = \{ \ z \in \mathbb{C} : 1 \leq |z| \leq 3\}.$$

Assume that

$$|f(z)| \leq 1, \ \text{ for } \ |z| = 1 \quad \text{and} \quad |f(z)| \leq 9, \ \text{ for } \ |z| = 3.$$

Show that $|f(z)| \leq |z|^2, \ \forall \ z \in \Delta$.

3. Let $U \subseteq \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ be a nonconstant holomorphic function such that $\mathrm{Re}(f(z)) \geq 0, \ \forall \ z \in U$. Prove that $\mathrm{Re}(f(z)) > 0, \ \forall \ z \in U$.

4. Let f be an entire function such that $|f(z)| \geq 1, \ \forall \ z \in \mathbb{C}$. Show that f is constant.

5. Let $f = u + iv$ be an entire function with $u^2 \leq v^2$. Show that f is constant.

6. Let f be an entire function such that

$$|f(z)| \leq Me^{aRe(z)}, \quad \forall\ z \in \mathbb{C},$$

where a, M positive real constants. Show that

$$f(z) = Ce^{az}, \quad \forall\ z \in \mathbb{C},$$

for some positive constant C.

7. Find the Laurent expansion of the function f around the point z_0 and on the "annulus" Δ, in each of the following cases:

(i) $f(z) = \dfrac{1}{z(z-1)}\ , \quad z_0 = -2, \quad \Delta = \{z \in \mathbb{C} : 2 < |z+2| < 3\}.$

(ii) $f(z) = \dfrac{\sin^2 z}{z}\ , \quad z_0 = 0, \quad \Delta = \mathbb{C} \setminus \{0\}.$

(iii) $f(z) = \sin\left(\dfrac{z}{1-z}\right)\ , \quad z_0 = 1, \quad \Delta = \mathbb{C} \setminus \{1\}.$

8. Let f be holomorphic and *bounded* on the punctured open disc $D(z_0, \delta) \setminus \{z_0\}$, $z_0 \in \mathbb{C}$, $\delta > 0$. Prove that z_0 is a removable singularity of f. [*Hint:* Use the integral folmulaes which give the Laurent's coefficients.]

9. Let f be holomorphic on the open disc $D(z_0, r)$ $(z_0 \in \mathbb{C},\ r > 0)$ with

$$f(z_0) = f'(z_0) = f''(z_0) = 0, \quad f'''(z_0) \neq 0.$$

(i) Show that there exists a holomorphic function φ on $D(z_0, r)$ satisfying

$$f(z) = (z - z_0)^3 \varphi(z), \quad \forall\, z \in\ D(z_0, r), \quad \varphi(z_0) \neq 0.$$

(ii) Assume that z_0 is the unique root of f on $D(z_0, r)$. Prove that

$$\operatorname{Res}\left(\frac{f'}{f}\ ,\ z_0\right) = 3.$$

10. (i) Let g be holomorphic on some neighborhood of 0 with $g(0) \neq 0$. Show that

$$\text{Res}\left(\frac{1}{z^3 g(z)}\ ,\ 0\right) = \frac{2[g'(0)]^2 - g''(0)g(0)}{2[g(0)]^3}\ .$$

(ii) Compute the integral

$$\int_\gamma \frac{dz}{z^2 \sin z}\ ,$$

where $\gamma(t) = 4e^{it},\ \ t \in [0, 2\pi]$.

11. (i) Find a holomorphic function $\varphi : \mathbb{C} \to \mathbb{C}$ such that

$$\sin z = (\pi - z)\varphi(z), \quad \forall z \in \mathbb{C}, \quad \varphi(\pi) = 1, \quad \varphi'(\pi) = 0.$$

(ii) Compute the integral

$$\int_\gamma \frac{e^z - 1}{\sin^2 z} dz\ ,$$

where $\gamma(t) = 4e^{it},\ \ t \in [0, 2\pi]$.

12. Compute the integral

$$\int_\gamma h(z) dz\ ,$$

where

$$h(z) = \frac{1}{1 - \cos z}\ +\ \overline{z} z^{12} \cos\left(1/z^3\right), \quad \gamma(t) = e^{it},\ \ t \in [0, 2\pi].$$

13. Compute the integral

$$\int_\gamma (1 - z^2) e^{1/z} dz\ ,$$

where $\gamma(t) = e^{it},\ \ t \in [0, 2\pi]$.

14. Using calculus of residues, prove:

(i) $$\int_0^{2\pi} \frac{\cos t}{2+\cos t} dt = 2\pi\left(1 - \frac{2}{\sqrt{3}}\right).$$

(ii) $$\int_0^{\pi} \frac{\sin^2 t}{a+\cos t} dt = \pi(a - \sqrt{a^2-1}), \quad a > 1.$$

15. Using calculus of residues, prove:

(i) $$\int_{-\infty}^{+\infty} \frac{x^4}{1+x^6} dx = \frac{2\pi}{3}.$$

(ii) $$\int_{-\infty}^{+\infty} \frac{\cos x}{(x^2+1)^2} dx = \pi/e.$$

(iii) $$\int_{-\infty}^{+\infty} \frac{x \sin x}{1+x^4} dx = -\pi \mathrm{Re}(\, a^2 e^{ia}), \text{ where } a = \frac{1+i}{\sqrt{2}}.$$