COMPLEX ANALYSIS
WORKSHEET 1
Instructor: G. Smyrlis

1. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(z) = \begin{cases} \dfrac{\overline{z}^{\,3}}{|z|^2}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

Show that Cauchy-Riemann conditions hold at $z_0 = 0$, whereas f fails to be differentiable at $z_0 = 0$.

2. Find the largest domain on which $\mathrm{Log}\left(\dfrac{1+z}{1-z}\right)$ is holomorphic.

3. Show that the function $f(z) = f(x + iy) = e^y \cos x + ie^y \sin x$ is nowhere differentiable in $\mathbb{C}$.

4. Detect the points where the function $f(z) = \overline{z}e^{-|z|^2}$ is differentiable. Then compute the derivative of f at each of these points.

5. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Show that the function $g(z) = \overline{f(\overline{z})}$ is holomorphic on $\mathbb{C}$ too.

6. Find the holomorphic function $f = u + iv : \mathbb{C} \to \mathbb{C}$ in each of the following cases:

   (i) $u(x, y) = -e^{-x} \sin y + \dfrac{y^2 - x^2}{2}$, $(x, y) \in \mathbb{R}^2$, $f(0) = 0$.

   (ii) $u(x, y) = 3x^2y - y^3 + e^{2y} \cos(2x)$, $(x, y) \in \mathbb{R}^2$, $f(0) = 1$.

7. Let $A \subseteq \mathbb{C}$ be a domain and $f = u + iv : A \to \mathbb{C}$ be holomorphic satisfying $u_x + v_y = 0$ in A. Show that there exist $c \in \mathbb{R}$, $d \in \mathbb{C}$ such that

$$f(z) = icz + d, \quad z \in A.$$

8. Set $f(z) = z^3$, $z_1 = \dfrac{-1 + i\sqrt{3}}{2}$, $z_2 = \dfrac{-1 - i\sqrt{3}}{2}$. Show that there is no z_0 in the segment $[z_1,\ z_2]$ such that

$$f(z_2) - f(z_1) = f'(z_0)(z_2 - z_1).$$

Conclude that Mean Value Theorem fails for complex functions.

9. Let $A \subseteq \mathbb{C}$ be open, $z_0 = x_0 + iy_0 \in A$ and $f = u + iv : A \to \mathbb{C}$. Assume that u, v have continuous partial derivatives on some neighborhood of (x_0, y_0) and also that the limit

$$\lim_{z \to z_0} \mathrm{Re}\left(\frac{f(z) - f(z_0)}{z - z_0}\right)$$

exists in $\mathbb{R}$. Show that f is differentiable at z_0.

10. Let $A \subseteq \mathbb{C}$ be a domain. Prove:

(i) If $f : A \to \mathbb{C}$ is a function such that both f, $\overline{f}$ are holomorphic, then f is constant.

(ii) If $f : A \to \mathbb{C}$ is holomorphic such that $|f|$ is constant, then f is constant.

(iii) If $f : A \to \mathbb{C}$ is a function such that both f^5, $\overline{f}^2$ are holomorphic, then f is constant.

11. Let $A \subseteq \mathbb{C}$ be a domain and $f : A \to \mathbb{C}$ be holomorphic. Prove:

(i) If $f(A)$ is contained in a straight line of the complex plane, then f is constant.

(ii) If $f(A)$ is contained in a circle of the complex plane, then f is constant.

12. (i) If x_0 is a negative real number, show that the limit $\lim_{w \to x_0} \mathrm{Log} w$ does not exist.

(*Hint*: Consider the sequences $|x_0|e^{i(\pi - 1/n)}$, $|x_0|e^{i(-\pi + 1/n)}$, $n \geq 1$.)

(ii) Show that there is no holomorphic function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that

$$(\mathrm{Re}\ f)(x, y) = \frac{1}{2} \ln(x^2 + y^2), \quad (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$$