$$f(z) = \frac{i}{2} \operatorname{Log}\left(\frac{z+i}{z-i}\right)$$

• <u>f ολόμορφη στο $\mathbb{C} \setminus [-i, i]$:</u>

$$\frac{z+i}{z-i} = \frac{(z+i)(\bar{z}+i)}{|z-i|^2} = \frac{(|z|^2-1) + 2i\operatorname{Re} z}{|z-i|^2}$$

$$\Rightarrow \quad \operatorname{Re}\left(\frac{z+i}{z-i}\right) = \frac{|z|^2-1}{|z-i|^2}, \quad \operatorname{Im}\left(\frac{z+i}{z-i}\right) = \frac{2\operatorname{Re} z}{|z-i|^2}.$$

f <u>όχι</u> διαφορ. στα σημεία z με $\begin{cases} |z|^2-1 \leq 0 \\ \operatorname{Re} z = 0 \end{cases}$

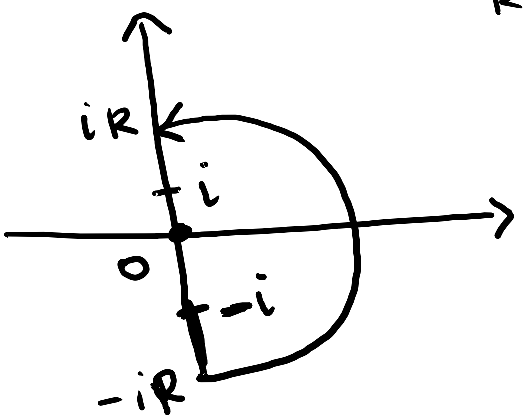
δηλ. στο $[-i, i]$.

$\cdot\ f'(z) = \dfrac{1}{z^2+1}, \quad \forall\ z \in U = \mathbb{C} \setminus [-i, i]$ (Πράξεις!)

Ορίζουμε

$$\underline{\operatorname{Arctan} z = \frac{i}{2} \operatorname{Log}\left(\frac{z+i}{z-i}\right), \quad z \in U.}$$

<u>Εφαρμογή:</u> $\displaystyle\int_{\gamma_R} \frac{dz}{z^2+1} = \, ?, \quad \gamma_R(t) = Re^{it}, \ t \in [-\pi/2, \pi/2], \ R > 1.$



$$\gamma_R^* \subset \mathbb{C} \setminus [-i, i]$$

$$\int_{\gamma_R} \frac{dz}{z^2+1} = \frac{i}{2}\left[\operatorname{Log}\left(\frac{iR+i}{iR-i}\right) - \operatorname{Log}\left(\frac{-iR+i}{-iR-i}\right)\right] =$$

$$= \frac{i}{2}\left[\operatorname{Log}\frac{R+1}{R-1} - \operatorname{Log}\frac{R-1}{R+1} \right]$$

$$= \frac{i}{2}\left(\ln\frac{R+1}{R-1} + \ln\frac{R+1}{R-1} \right) = i \ln\frac{R+1}{R-1}.$$