ΑΣΚΗΣΕΙΣ (26/05/2021).

ΦΥΛΛΑΔΙΟ 4

(12) $\int_\gamma h(z)\,dz = ?$, $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$

$h(z) = f(z) + g(z)$, $f(z) = \dfrac{1}{1-\cos z}$, $g(z) = \bar{z} z^{12} \cos\left(\dfrac{1}{z^3}\right)$

Λύση: $\int_\gamma h = \int_\gamma f + \int_\gamma g$.

• $\int_\gamma f$

Ανώμαλα σημεία της f: $2k\pi$, $k \in \mathbb{Z}$,

Μόνο $0 \in \operatorname{int}\gamma^* \Rightarrow \int_\gamma f = 2\pi i \operatorname{Res}(f, 0)$.

$$f(z) = \frac{1}{1-\left(1-z^2/2! + z^4/4! - \cdots\right)} = \frac{1}{\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots}$$

$$= \frac{1}{z^2} \cdot \frac{1}{\underbrace{\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots}_{\varphi(z)}} = \frac{\psi(z)}{z^2}, \quad \psi = 1/\varphi$$

φ ολόμορφη στο $\mathbb{C}$ κ' $\underline{\varphi(0) = 1/2 \neq 0}$, $\varphi'(0) = 0$

$\left[\frac{\varphi^{(n)}(0)}{n!} = \text{συντελ}(z^n)\right]$ $\Longrightarrow$ ψ ολόμορφη σε περιοχή του 0

κ' $\psi(0) \neq 0$

$\Longrightarrow$

$\Longrightarrow$ $0 =$ πόλος τάξης 2

$$\Longrightarrow \operatorname{Res}(f,0) = \lim_{z\to 0} \left[z^2 f(z)\right]' = \psi'(0) =$$

$$\overset{\psi = 1/\varphi}{=} -\frac{\varphi'(0)}{\varphi(0)^2} = 0.$$

Άρα, $\int_\gamma f(z)\,dz = 2\pi i \cdot 0 = 0.$

$$g(z) = \bar{z}\, z^{12} \cos\left(\frac{1}{z^3}\right) = \frac{1}{z}\, z^{12} \cos\left(\frac{1}{z^3}\right) = z^{11} \cos\left(\frac{1}{z^3}\right),$$

$$\forall\, z \in \gamma^* \ (\ |z| = 1).$$

$$\int_\gamma g = \int_\gamma \theta(z)\,dz, \qquad \theta(z) = z^{11} \cos\left(\frac{1}{z^3}\right) =$$

$$= z^{11}\left[1 - \left(\frac{1}{z^3}\right)^2 \frac{1}{2!} + \left(\frac{1}{z^3}\right)^4 \cdot \frac{1}{4!} - \left(\frac{1}{z^3}\right)^6 \cdot \frac{1}{6!} + \cdots\right]$$

$$= z^{11}\left(1 - \frac{1}{2!\,z^6} + \frac{1}{4!\,z^{12}} - \frac{1}{6!\,z^{18}} + \cdots\right)$$

$$= z^{11} - \frac{1}{2!} z^5 + \frac{1}{4!} \frac{1}{z} - \cdots$$

$$\Longrightarrow \operatorname{Res}(\theta(z), 0) = 1/4! \Longrightarrow \int_\gamma \theta(z)\,dz = \frac{2\pi i}{4!} =$$

$$= \frac{2\pi i}{24} = \pi i/12.$$

Αρχικό ολοκλ. $=$

$$= 0 + \pi i/12 = \pi i/12.$$

(13) $I = \int_\gamma f(z)\,dz = ?$, $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$

$f(z) = (1 - z^2)e^{1/z}$

ΛΥΣΗ:

Ανώμαλα σημεία: $0 \in \text{int}\,\gamma^*$

$\Rightarrow I = 2\pi i \operatorname{Res}(f, 0)$.

$$f(z) = (1 - z^2)\left(1 + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \frac{1}{4!}\frac{1}{z^4} + \frac{1}{5!}\frac{1}{z^5} + \cdots\right)$$

$$= \cdots + \frac{1}{z}\underbrace{\left(1 \cdot \frac{1}{1!} - \frac{1}{3!}\right)} + \cdots$$

$\operatorname{Res}(f, 0) = 1 - \frac{1}{6} = 5/6$

$\Rightarrow \int_\gamma f = 2\pi i \cdot 5/6 = \boxed{5\pi i/3.}$

(14) (ii) Να δ.ο. $I = \int_0^{\pi} \frac{\sin^2 t}{a+\cos t}\,dt = \pi\left(a-\sqrt{a^2-1}\right)$, $a>1$.

ΛΥΣΗ:

$$\int_{\pi}^{2\pi} \frac{\sin^2 t}{a+\cos t}\,dt \overset{x=2\pi-t}{=} \int_{\pi}^{0} \frac{\sin^2 x}{a+\cos(2\pi-x)}(-dx)$$

$$= \int_0^{\pi} \frac{\sin^2 x}{a+\cos x}\,dx = I.$$

$$J = \int_0^{2\pi} \frac{\sin^2 t}{a+\cos t}\,dt = \int_0^{\pi} + \int_{\pi}^{2\pi} = 2I$$

$$\Rightarrow \quad I = \frac{1}{2} J.$$

$$J = \int_\gamma \frac{\left(\frac{z^2-1}{2iz}\right)^2}{a + \frac{z^2+1}{2z}} \frac{dz}{iz} =$$

$$= \int_\gamma \frac{\frac{(z^2-1)^2}{-4z^2}}{aiz + \frac{i(z^2+1)}{2}} =$$

$$= -\frac{1}{i} \int_\gamma \frac{(z^2-1)^2}{4az^3 + 2z^2(z^2+1)} dz =$$

$$= \frac{i}{2} \int_\gamma \boxed{\frac{(z^2-1)^2}{z^2(z^2+2az+1)}}\, dz .$$

$\underline{\text{Ρίζες του } z^2+2az+1}$: $\overset{f(z)}{}$ $\rho_1 = -a + \sqrt{a^2-1} = \rho$, $\rho_2 = 1/\rho$

$$|\rho| = a - \sqrt{a^2-1} < 1 < \left|\frac{1}{\rho}\right| \Rightarrow \underline{\rho \in \operatorname{int}\gamma^*}, \ \frac{1}{\rho} \notin \operatorname{int}\gamma^*$$

$$\Rightarrow \int_\gamma f(z)\,dz = 2\pi i \left[\operatorname{Res}(f,0) + \operatorname{Res}(f,\rho) \right].$$

- $\underline{\mathrm{Res}(f,0)}$ $\qquad f(z) = \dfrac{A(z)}{B(z)} \cdot \dfrac{1}{z^2}$,

$A(z) = (z^2-1)^2$, $\quad B(z) = z^2 + 2az + 1$

$A(0) = 1$, $A'(0) = 0$, $\quad B(0) = 1$, $\quad B'(0) = 2a$

$$\mathrm{Res}(f,0) = \lim_{z\to 0} \left[z^2 f(z)\right]' = \frac{A'(0)B(0) - A(0)B'(0)}{B(0)^2}$$

$$= -2a$$

- $\underline{\mathrm{Res}(f,\rho)}$ $\qquad f(z) = \dfrac{\varphi(z)}{B(z)}$, $\qquad \varphi(z) = \dfrac{(z^2-1)^2}{z^2}$

$$= \left(z - \frac{1}{z}\right)^2$$

$$\mathrm{Res}(f,\rho) = \frac{\varphi(\rho)}{B'(\rho)}$$

$$\varphi(p) = \left(p - \frac{1}{p}\right)^2 = (p_1 - p_2)^2 = (p_1 + p_2)^2 - 4p_1p_2$$

$$= (-2a)^2 - 4 \cdot 1 = 4(a^2-1),$$

$$B'(p) = 2p + 2a = 2(p+a) = 2\sqrt{a^2-1}$$

$$\Longrightarrow \quad \mathrm{Res}(f,p) = \frac{4(a^2-1)}{2\sqrt{a^2-1}} = 2\sqrt{a^2-1}$$

Άρα: $$\int_\gamma f(z)\,dz = 2\pi i\left(-2a + 2\sqrt{a^2-1}\right)$$

$$= 4\pi i\left(-a + \sqrt{a^2-1}\right)$$

$$\Longrightarrow \quad J = \frac{i}{2}\int_\gamma f(z)\,dz = 2\pi\left(a - \sqrt{a^2-1}\right)$$

$$\Longrightarrow \quad I = J/2 = \underline{\pi\left(a - \sqrt{a^2-1}\right)}.$$

15 **(i)** $$I=\int_{-\infty}^{+\infty}\frac{x^4}{1+x^6}\,dx = 2\pi/3 .$$

ΛΥΣΗ

$P(x)=x^4$, $Q(x)=1+x^6$, $Q(x)\neq 0, \forall x\in\mathbb{R}$

$$\deg Q - \deg P = 2$$

$f(z)=\dfrac{z^4}{1+z^6}$ Ρίζες του Q:

$$\alpha = e^{i\pi/6} = \frac{\sqrt{3}+i}{2}, \quad i^6=(i^2)^3=-1$$

$e^{i\pi}=-1$ $\underline{\alpha}$, $\underline{i}$

$\{\pm a, \ \pm \bar{a}, \ \pm i\}$ ol eijes en $\mathbb{Q}$

$\boxed{a, \ -\bar{a}, \ i}$ (¡xon Im ≥ 0!)

$$I = 2\pi i \left[\operatorname{Res}(f, a) + \operatorname{Res}(f, -\bar{a}) + \operatorname{Res}(f, i) \right]$$

$\forall p \in \{a, -\bar{a}, i\}$,

$$\operatorname{Res}(f, p) = \left.\frac{z^4}{6z^5}\right|_{z=p} = \frac{1}{6}\frac{1}{p} = \frac{\bar{p}}{6}$$

$$\Longrightarrow I = \frac{2\pi i}{6}\left(\bar{a} + \overline{(-\bar{a})} + \bar{i} \right) =$$

$$= \frac{\pi i}{3}\left(\bar{a} - a - i\right) = \frac{\pi i}{3}\left(-2i\,\mathrm{Im}\,a - i\right)$$

$$= \frac{\pi i}{3}(-i)\left(2\,\mathrm{Im}\,a + 1\right)$$

$$= \frac{\pi}{3}\left(2 \cdot \frac{1}{2} + 1\right) = \frac{2\pi}{3}.$$