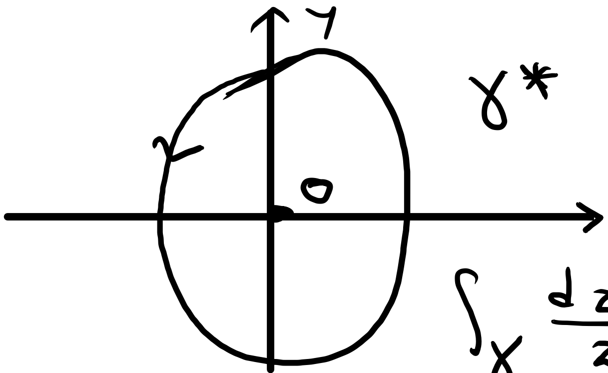
ΦΥΛΛΑΔΙΟ 3

(3) Να δ.ο. $\int_0^{2\pi} \frac{dt}{4\cos^2 t + 9\sin^2 t} = \pi/3$, ολοκληρώνοντας την $1/z$ πάνω στην έλλειψη $\gamma(t) = 2\cos t + 3i\sin t, t\in[0,2\pi]$.



$\int_\gamma \frac{dz}{z} \overset{\text{O.T.}}{\underset{\text{Cauchy}}{=}} 2\pi i$

Ταυτόχρονα,

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)}dt = \int_0^{2\pi} \frac{\gamma'(t)\overline{\gamma(t)}}{|\gamma(t)|^2}dt,$$

$|\gamma(t)|^2 = 4\cos^2 t + 9\sin^2 t,$

$$\gamma'(t)\overline{\gamma(t)} = (-2\sin t + 3i\cos t)(2\cos t - 3i\sin t)$$

$$= -4\sin t\cos t + 6i\sin^2 t + 6i\cos^2 t + 9\sin t\cos t$$

$$= 5\sin t\cos t + 6i$$

$$\Rightarrow \int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{5\sin t\cos t + 6i}{4\cos^2 t + 9\sin^2 t} =$$

$$= \underbrace{\qquad\qquad}_{A} + 6i \underbrace{\boxed{\int_0^{2\pi} \frac{dt}{4\cos^2 t + 9\sin^2 t}}}_{J}$$

$$\Rightarrow 2\pi i = A + 6iJ \quad \Rightarrow \quad J = 2\pi/6 = \pi/3,$$

(11) (ii) Να βρείτε το ανάπτυγμα Taylor γύρω από το $z_0 = \pi$, της $f(z) = \cos^2 z$.

Λύση: $f(z) = \dfrac{1+\cos(2z)}{2} = \dfrac{1}{2} + \dfrac{1}{2}\cos(2z)$

$w = z - \pi \Rightarrow z = \pi + w$

$$\Rightarrow \cos(2z) = \cos(2\pi + 2w) = \cos(2w) =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2w)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} w^{2n} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} (z-\pi)^{2n}$$

κ.λ.π.

(15) $$f(z) = \frac{(1-\cos z)^2}{(e^z - 1 - z)\sin^2 z}, \quad \lim_{z\to 0} f(z) = ?$$

Çözüm:

$$f(z) = \frac{\left(z^2/2! - z^4/4! + z^6/6! - \cdots\right)^2}{\left(z^2/2! + z^3/3! + z^4/4! + \cdots\right)\left(z - z^3/3! + z^5/5! - \cdots\right)^2}$$

$$= \frac{\cancel{z^4}\left(\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots\right)^2}{\cancel{z^4}\left(\frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots\right)\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)^2}$$

$$\Rightarrow \lim_{z\to 0} f(z) = \frac{(1/2!)^2}{1/2! \cdot 1^2} = 1/2.$$

16 Έστω $f:\mathbb{C}\to\mathbb{C}$ ακέραια με $f(\mathbb{R})\subset\mathbb{R}$.

(i) $\forall n\in\mathbb{N}$, $f^{(n)}(\mathbb{R})\subset\mathbb{R}$.

(ii) $f(\bar{z})=\overline{f(z)}$, $\forall z\in\mathbb{C}$.

Λύση:

Ισχυρισμός: Εάν $g:\mathbb{C}\to\mathbb{C}$ ακέραια με $g(\mathbb{R})\subseteq\mathbb{R}$, τότε κ' $g'(\mathbb{R})\subseteq\mathbb{R}$.

Απόδειξη του Ισχυρισμού:

$\longrightarrow$

Έστω $x_0 \in \mathbb{R}$.

Επειδή g διαφορ. στο x_0, ισχύει

$$g'(x_0) = \lim_{z \to x_0} \frac{g(z) - g(x_0)}{z - x_0}$$

$$\Rightarrow \quad g'(x_0) = \lim_{\substack{z \to x_0 \\ z \in \mathbb{R}}} \frac{g(z) - g(x_0)}{z - x_0} \in \mathbb{R}$$

αφού $g(z) \in \mathbb{R}$, $\forall z \in \mathbb{R}$ κ' $g(x_0) \in \mathbb{R}$.

(i) Η απόδειξη θα γίνει με επαγωγή στο n.

- Για $n=0$, ισχύει, αφού $f(\mathbb{R}) \subseteq \mathbb{R}$.
- Υποθέτουμε ότι ισχύει για κάποιο $n \in \mathbb{N}$, δηλ.

$$f^{(n)}(\mathbb{R}) \subseteq \mathbb{R}.$$

Εφαρμόζοντας τον Ισχυρισμό για "g" $= f^{(n)}$ παίρνουμε

$$g'(\mathbb{R}) \subseteq \mathbb{R} \Rightarrow f^{(n+1)}(\mathbb{R}) \subseteq \mathbb{R},$$

άρα ισχύει κ' για $n+1$.

(ii) Θ. Taylor $\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \ \forall z \in \mathbb{C}.$

λόγω του (i), $f^{(n)}(0) \in \mathbb{R}, \ \forall n \in \mathbb{N}$, οπότε

$\forall z \in \mathbb{C}$,

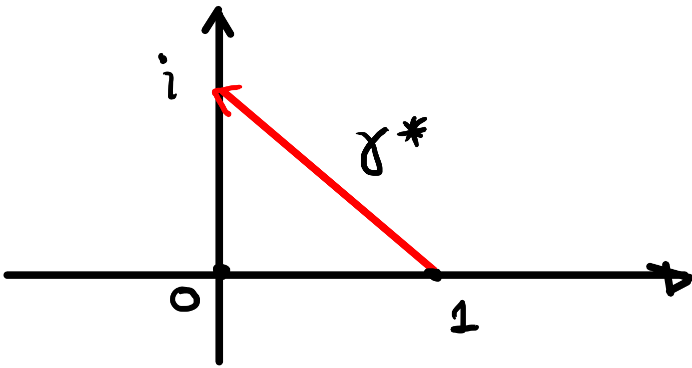
$$\overline{f(z)} = \overline{\left[\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n\right]} = \sum_{n=0}^{\infty} \overline{\frac{f^{(n)}(0)}{n!}} \, \bar{z}^n =$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \bar{z}^n = f(\bar{z}).$$

ΦΥΛΛΑΔΙΟ 2

① (iv) $\int_\gamma \frac{\operatorname{Log} z}{z}\, dz = ?$, $\quad \gamma = [1, i]$

Λύση: $\gamma^* \subset U = \mathbb{C} \setminus (-\infty, 0]$



κ' η $F(z) = \frac{1}{2} \operatorname{Log}^2 z$

είναι ολόμορφη στο U

με $F'(z) = \frac{\operatorname{Log} z}{z}$, $\forall z \in U$

$$\Rightarrow \int_\gamma \frac{\operatorname{Log} z}{z}\, dz = F(i) - F(1)$$

$$= \frac{1}{2} \operatorname{Log}^2 i = \frac{1}{2} \left[\ln|i| + i \operatorname{Arg}(i)\right]^2$$

$$= \frac{1}{2}\left(\frac{\pi i}{2}\right)^2 = -\frac{\pi^2}{8}.$$

(vi) $\int_\gamma \overline{z^2 e^z}\, dz = ?$, $\quad \gamma(t) = e^{it}$, $t \in [0,\pi]$.

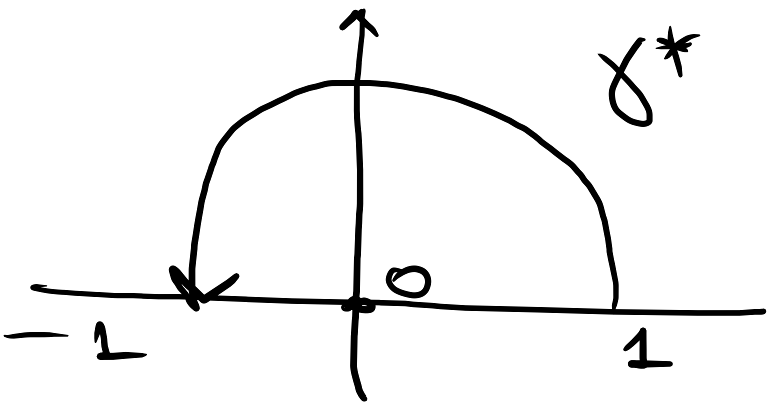
$\forall z \in \gamma^*$, $|z| = 1 \Rightarrow \bar{z} = 1/z$

$$\Rightarrow \overline{z^2 e^z} = \bar{z}^2 e^{\bar{z}} = \frac{1}{z^2} e^{1/z} = -\left(e^{1/z}\right)'$$

$F(z) = -e^{1/z}$, ολόμορφη στο $U = \mathbb{C}\setminus\{0\} \supset \gamma^*$

κ' $F'(z) = \frac{1}{z^2} e^{1/z}$, $\forall z \in U$.

$$\Rightarrow \int_\gamma \overline{z^2 e^z}\, dz = \int_\gamma \frac{1}{z^2} e^{1/z}\, dz = -e^{1/z}\Big|_{z=1}^{z=-1}$$



$$= -\left(e^{-1} - e\right)$$

$$= e - \frac{1}{e} .$$

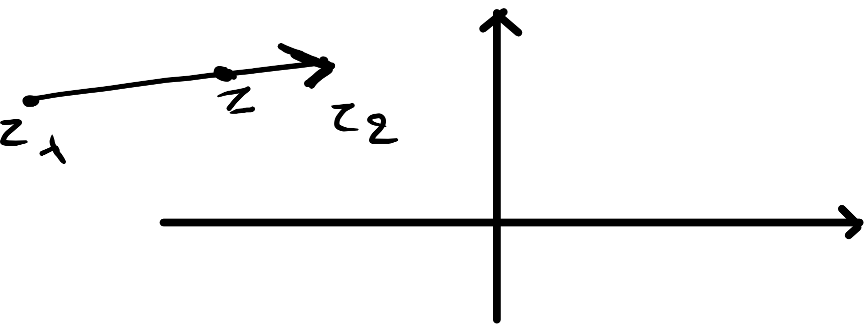
⑤ Έστω $z_1, z_2 \in \mathbb{C}$ με $\underline{Re(z_1) \le 0}$, $\underline{Re(z_2) \le 0}$,

ν.δ.ο. $$|e^{z_1} - e^{z_2}| \le |z_1 - z_2|.$$

<u>Λύση:</u>

$$\int_{[z_1, z_2]} e^z dz = \int_{[z_1, z_2]} (e^z)' dz = e^z \Big|_{z_1}^{z_2}$$

$$= e^{z_2} - e^{z_1}. \quad (\dagger)$$



$\forall z \in [z_1, z_2]$, $Re(z) \le 0$

$$\Longrightarrow |e^z| = e^{Re(z)} \le 1$$

ML-ανισ.

$$\Longrightarrow \quad \left| \int_{[z_1, z_2]} e^z \, dz \right| \leq 1 \cdot |z_1 - z_2| = |z_1 - z_2|$$

$$\Longrightarrow \quad |e^{z_2} - e^{z_1}| \leq |z_1 - z_2| .$$

(†) $\forall z \in [z_1, z_2]$ τότε $z = (1-t) z_1 + t z_2$ για κάποιο $t \in [0,1]$

$$\Longrightarrow \operatorname{Re}(z) = \operatorname{Re}[(1-t) z_1] + \operatorname{Re}(t z_2)$$

$$= (1-t) \underline{\operatorname{Re} z_1} + t \, \underline{\operatorname{Re} z_2} \leq 0 + 0 = 0 .$$

(8) (i) Εάν $\varphi : [a,b] \to \mathbb{C}$ διαφορ. να δ.ο.

$$\frac{d}{dt}\left[|\varphi(t)|^2\right] = 2\mathrm{Re}\left[\varphi'(t)\overline{\varphi(t)}\right], \forall t\in[a,b].$$

Λύση: (i)

$$\varphi(t) = \varphi_1(t) + i\varphi_2(t), \quad t\in[a,b],$$

με $\varphi_1, \varphi_2 : [a,b] \to \mathbb{R}$ διαφορίσιμες.

$$|\varphi(t)|^2 = \varphi_1(t)^2 + \varphi_2(t)^2$$

$$\Longrightarrow \frac{d}{dt}\left[|\varphi(t)|^2\right] = 2\varphi_1(t)\varphi_1'(t) + 2\varphi_2(t)\varphi_2'(t).$$

Επιπλέον,

$$\varphi'(t)\overline{\varphi(t)} = [\varphi_1'(t) + i\varphi_2'(t)]\cdot[\varphi_1(t) - i\varphi_2(t)]$$

$$= [(\varphi_1'\varphi_1 + \varphi_2\varphi_2') + i(\varphi_2'\varphi_1 - \varphi_1'\varphi_2)](t),$$

$\forall t \in [a,b]$

$$\Rightarrow 2\mathrm{Re}\,[\varphi'(t)\overline{\varphi(t)}] = 2(\varphi_1'\varphi_1 + \varphi_2\varphi_2')(t) = \frac{d}{dt}\left[|\varphi(t)|^2\right].$$

(ii) Έστω $f: U \to \mathbb{C}$ ολόμορφη (U ανοικτό) κ' γ απλή, κλειστή λεία με $\gamma^* \subset U$. Να δ-ο. $\int_\gamma f'(z)\overline{f(z)}\,dz$ φανταστ.

Λύση: Έστω $\gamma : [a,b] \to \mathbb{C}$ λεία, κλειστή.

$$\int_\gamma f'(z)\,\overline{f(z)}\,dz = \int_a^b f'(\gamma(t))\,\overline{f(\gamma(t))}\,\gamma'(t)\,dt$$

$$= \int_a^b \frac{d}{dt}\left[f(\gamma(t))\right] \cdot \overline{f(\gamma(t))}\,dt \overset{\varphi(t)=f(\gamma(t))}{=\!=}$$

$$= \int_a^b \varphi'(t)\,\overline{\varphi(t)}\,dt$$

$$\Longrightarrow \mathrm{Re}\left(\int_\gamma f'\bar{f}\right) = \int_a^b \mathrm{Re}\left[\varphi'(t)\overline{\varphi(t)}\right]dt \overset{(i)}{=}$$

$$= \frac{1}{2} \int_a^b \frac{d}{dt} \left[|\varphi(t)|^2 \right] dt =$$

$$= \frac{1}{2} \left(|\varphi(b)|^2 - |\varphi(a)|^2 \right)$$

$$= \frac{1}{2} \left(|f(\gamma(b))|^2 - |f(\gamma(a))|^2 \right) \underset{(\gamma(a)=\gamma(b)!)}{\overset{\gamma \text{ κλειστή}}{=}}$$

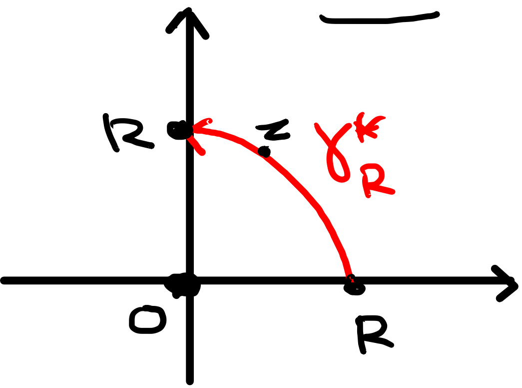
$$= 0 \; !!$$

④ Να δ.ο.

$$\lim_{R\to+\infty}\int_{\gamma_R}\frac{e^{iz^2}}{1+z^2}\,dz=0, \quad \text{όπου}$$

$\gamma_R(t)=Re^{it}$, $t\in[0,\pi/2]$, $R>0$.

ΛΥΣΗ:



Έστω $z=x+iy\in\gamma^*_R$

$\Rightarrow |z|=R$ και

$iz^2=i(x^2-y^2+2ixy)$

$=-2xy+i(x^2-y^2)$

$\Rightarrow |e^{iz^2}|=e^{Re(iz^2)}=e^{-2xy}\le 1$

Επιπλέον, $\forall z \in \gamma_R^*$,

$$|z^2+1| \geq |z|^2 - 1 = R^2 - 1$$

$$\Longrightarrow \forall R > 1, \quad \forall z \in \gamma_R^*, \quad \frac{1}{|z^2+1|} \leq \frac{1}{R^2-1}$$

$$\Longrightarrow \left| \frac{e^{iz^2}}{1+z^2} \right| \leq \frac{1}{R^2-1} \overset{ML}{\Longrightarrow} \left| \int_{\gamma_R} \frac{e^{iz^2}}{1+z^2}\, dz \right| \leq$$

$$\leq \frac{1}{R^2-1} \frac{\pi R}{2} \xrightarrow{R \to +\infty} 0$$

(7) $\int_{\gamma_r} \mathrm{Re}\, z\, dz = ?$, $\gamma_r(t) = re^{it}$, $t \in [0, 2\pi]$ $(r > 0)$.

Να δ.ο. η $\mathrm{Re}\, z$ δεν έχει παράγουσα σε κανένα ανοικτό $U \subset \mathbb{C}$ με $0 \in U$.

<u>Λύση:</u> $\int_{\gamma_r} \mathrm{Re}\, z\, dz = \int_0^{2\pi} \mathrm{Re}(re^{it})\, ire^{it}\, dt =$

$$= \int_0^{2\pi} (r\cos t)\; ir\,(\cos t + i\sin t)\,dt$$

$$= ir^2 \left(\int_0^{2\pi} \cos^2 t\, dt + i \underbrace{\int_0^{2\pi} \cos t \sin t\, dt}_{0} \right)$$

$$= ir^2 \int_0^{2\pi} \frac{1+\cos(2t)}{2}\,dt =$$

$$= i\pi r^2 \neq 0$$

Έστω U ανοικτό $\ni 0$. Υποθέτουμε ότι η $f(z) = \operatorname{Re} z$ έχει παράγουσα στο U. $\exists\, r > 0 \mid \gamma_r^* \subset U$ όπου $\gamma_r(t) = re^{it}$. Τότε, γ_r κλειστή $\Rightarrow \int_{\gamma_r} f(z)\,dz = 0$

(Άτοπο!)